What is a graph limit?

Jan Volec

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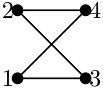

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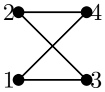

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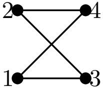

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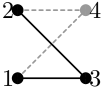

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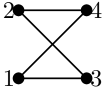

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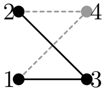

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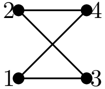

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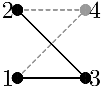

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- mainly for convenience, assume $v(G_n) \to \infty$

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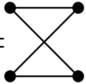
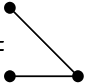
$G_n =$  $F =$ 

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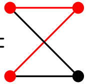
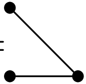
$G_n =$  $F =$ 

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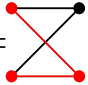
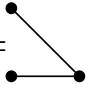
$G_n =$  $F =$ 

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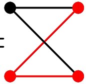
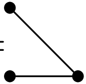
$G_n =$  $F =$ 

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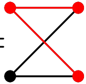
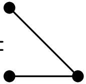
$G_n =$  $F =$ 

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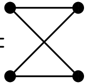
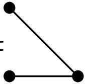
$G_n =$  $F =$  $t(G_n, F) = 1$

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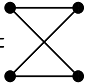
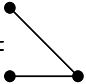
$G_n =$  $F =$  $t(G_n, F) = 1$

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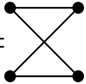
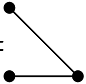
$G_n =$  $F =$  $t(G_n, F) = 1$

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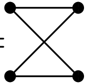
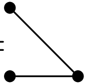
$G_n =$  $F =$  $t(G_n, F) = 1$

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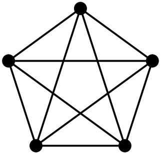 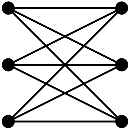 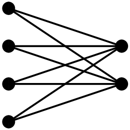 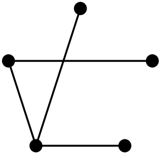

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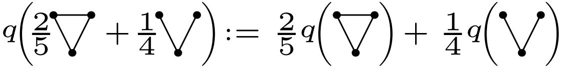

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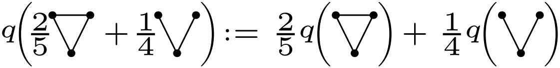

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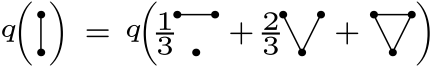

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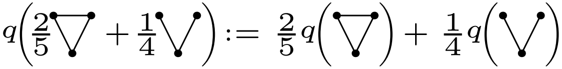

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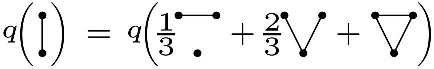

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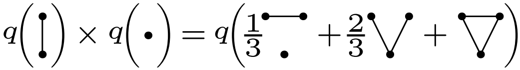

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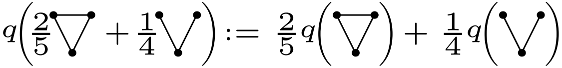

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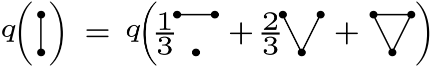

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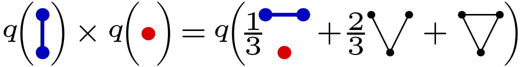

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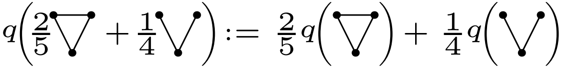

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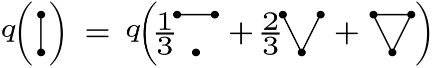

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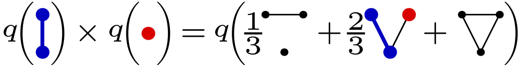

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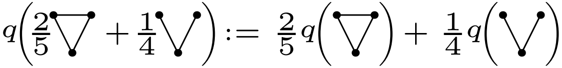

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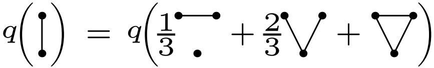

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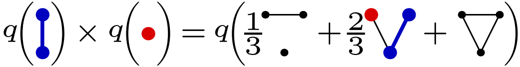

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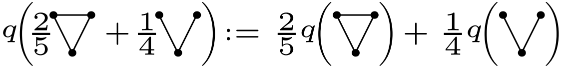
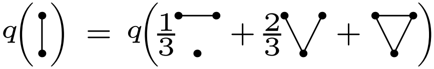


- lifting up in q:



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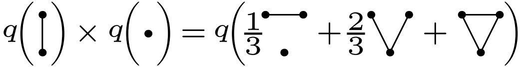

Graph limits – counting point of view

$\mathcal{F} := \{\text{all finite graphs } F\}$ $\quad (G_n)$ convergent sequence of graphs
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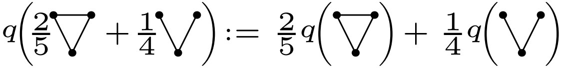

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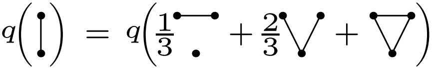

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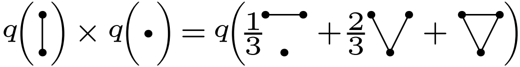

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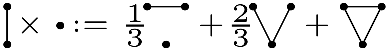

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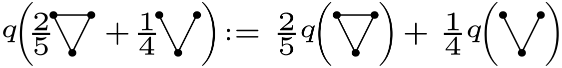

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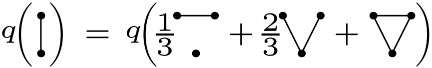

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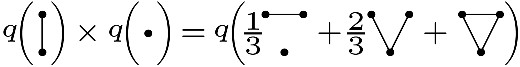

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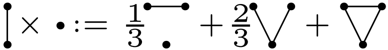

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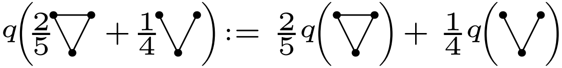

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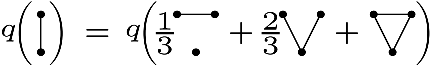

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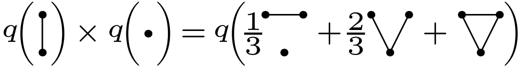

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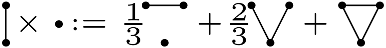

- q is homomorphism $\mathbb{R}\mathcal{F} \to \mathbb{R}$ with $q(F) \geq 0$ for all $F \in \mathcal{F}$
- for every such q there is convergent (G_n) so that $(G_n) \to q$

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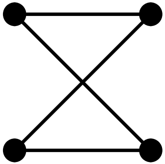

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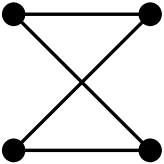

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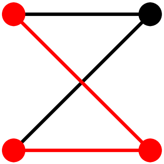

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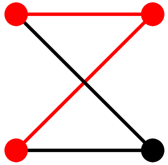

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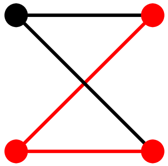

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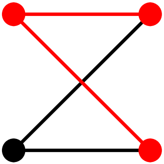

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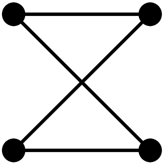

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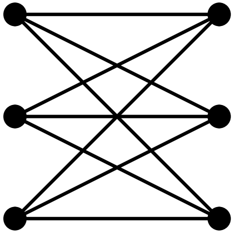

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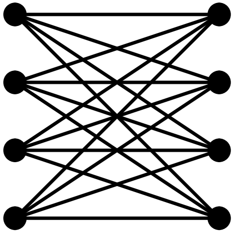

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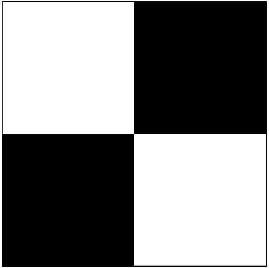

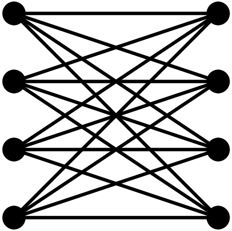

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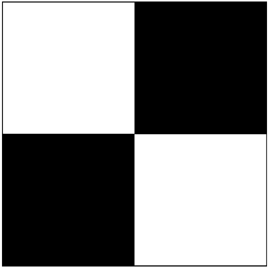

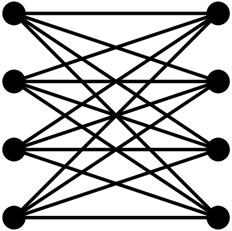

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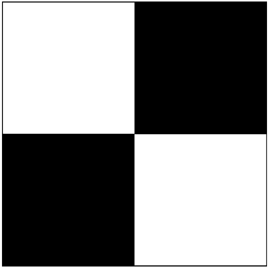

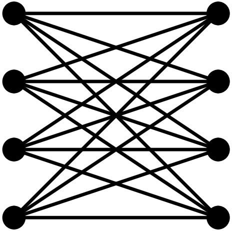

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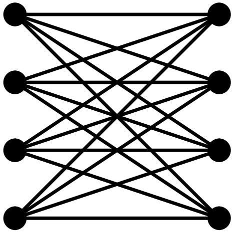

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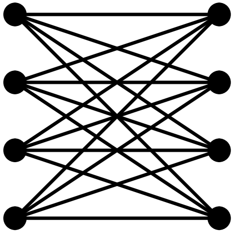

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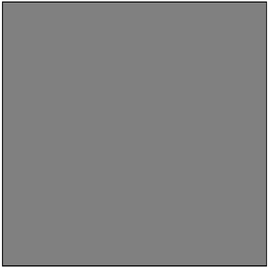

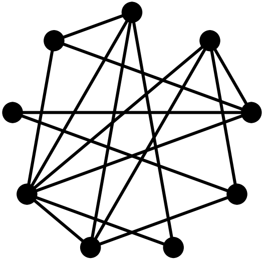

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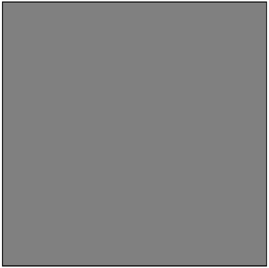

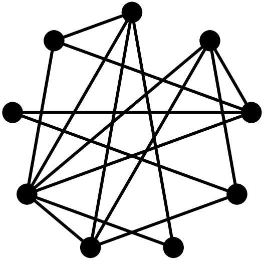

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we group "similarly-behaving" vertices together via Regularity Lemma

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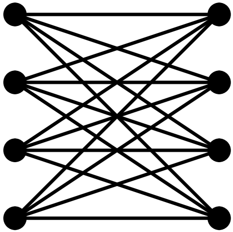

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we group "similarly-behaving" vertices together via Regularity Lemma

Graph limits – analytical point of view

Fix F so that $v(F) = k$, $t(G, F) = \#$ of copies of F in G / $\binom{v(G)}{k}$

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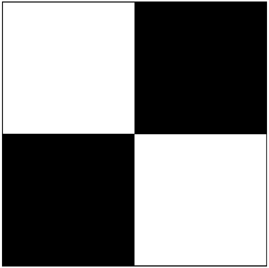

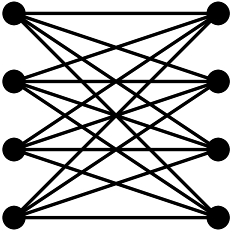

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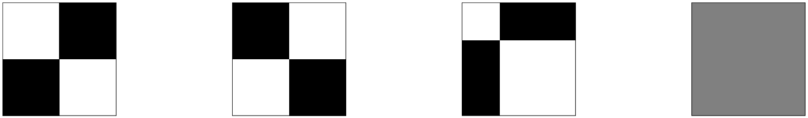

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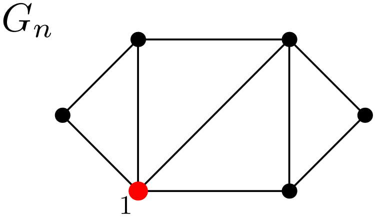
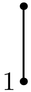


$F^1 =$ 

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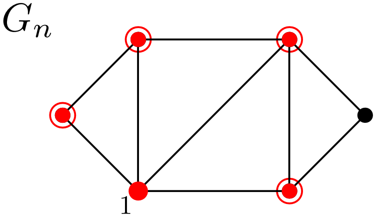
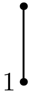


$F^1 =$ 

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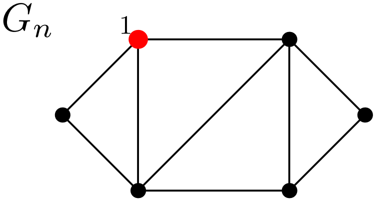
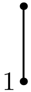


$F^1 =$ 

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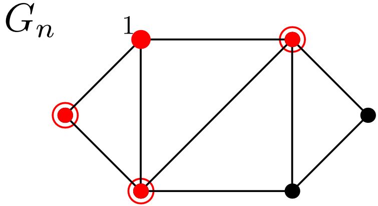

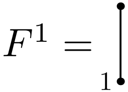

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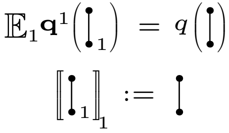

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$$\left[\!\!\left[\left(\sum_{F \in \mathcal{F}^1} \alpha_F F^1\right)^2\right]\!\!\right]_1 \geq \left(\left[\!\!\left[\sum_{F \in \mathcal{F}^1} \alpha_F F^1\right]\!\!\right]\right)^2 \geq 0$$

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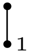
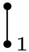
$$q\left(\left[\!\left[\left(\right.\right.\right.\right.$$  $$\left.\left.\left.\left. \right)^2 \right]\!\right]_1 \right) \geq q\left(\left[\!\left[\right.\right.\right.$$  $$\left.\left.\left. \right]\!\right]_1 \right)^2$$

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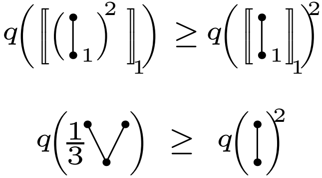

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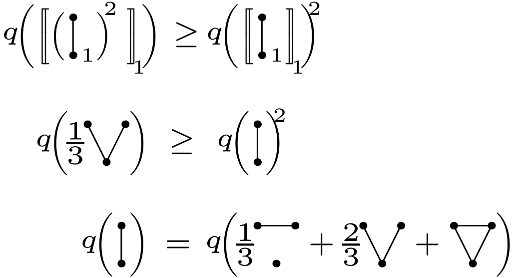

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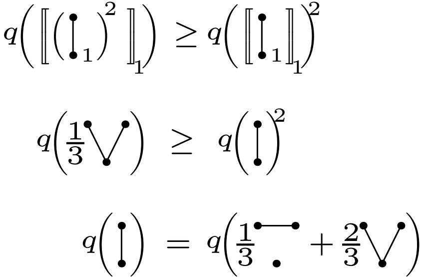

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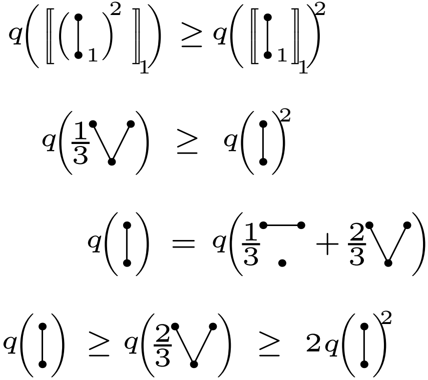

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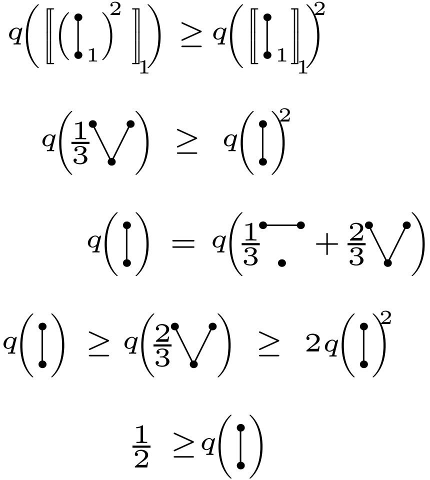

Thank you for your attention!