MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #$\emptyset$: Just a bunch of problems and exercises

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Let $G = (V, E)$ be a simple graph and let $\delta(G) := \min_{v \in V} \deg_G(v)$.

a) G contains a path of length δ.

b) If $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta + 1$.

$\star$) If G is connected and has $|V| = n$ vertices, then G contains a path of length $\min\{2\delta, n-1\}$.

Recall that a complement of a simple graph $G = (V, E)$ is the graph $\overline{G} = (V, \binom{V}{2} \setminus E)$. A simple graph G is called *self-complementary* if G is isomorphic to its complement $\overline{G}$.

a) Find a self-complementary graph G on at least two vertices.

b) Show that if G is an n-vertex self-complementary graph, then $n = 4k$ or $n = 4k + 1$ for some integer $k \geq 0$.

$\star$) Construct a self-complementary graph on n vertices for infinitely many values of n.

Let $k \geq 2$ be an integer.

a) Show that if $G = (V, E)$ is a k-connected simple graph, then for any k-vertex subset $U \subseteq V$ there exists a cycle C in G such that $U \subseteq V(C)$.

b) Construct a k-connected simple graph $G = (V, E)$ that contains a $(k+1)$-vertex subset $U \subseteq V$ such that no cycle C in G satisfies $U \subseteq V(C)$.

c) Show that if $G = (V, E)$ is a k-connected simple graph, then for any $(k + 1)$-vertex subset $U \subseteq V$ there exists a path P in G such that $U \subseteq V(P)$.

d) Construct a k-connected simple graph $G = (V, E)$ that contains a $(k+2)$-vertex subset $U \subseteq V$ such that no path P in G satisfies $U \subseteq V(P)$.

Let $G = (V, E)$ be a simple graph. Recall an edge $e \in E$ is called *a cut-edge* if the number of connected components of the graph $G - e$ is strictly larger than the number of connected components of G. Also recall that a graph G is called k-regular if every vertex has degree exactly k.
Prove that if G is a $2k$-regular graph, then G contains no cut-edge.

Let $G = (V, E)$ be a multigraph without loops such that $\ell > 0$ vertices have an odd degree.

a) Recall that ℓ must be even.

b) Show that if G is connected, then there exists tours $T_1, \dots, T_\ell$ in G such that every edge $e \in E$ is contained in exactly one of the tours.

A Hamiltonian path in a graph $G = (V, E)$ is a path in G that contains all the vertices, i.e., a path of length $|V| - 1$. Let $G = (V, E)$ be a simple graph and let $\delta(G) := \min_{v \in V} \deg_G(v)$.

a) If $\delta(G) \geq n/2$, then G contains a Hamiltonian cycle.

b) If $\delta(G) \geq (n-1)/2$, then G contains a Hamiltonian path.

$\star$) If $\delta(G) \geq n/2 + 1$ and $u, w \in V$ are any two vertices of G, then G contains a path from u to w of length $|V| - 1$, i.e., a Hamiltonian path with the endpoints u and w.

Let $G = (V, E)$ be a simple bipartite graph with parts A and B such that $|A| = |B| = n$.

a) If the minimum degree $\delta(G) \geq n/2$, then G contains a perfect matching.

b) Show that the minimum degree condition cannot be improved, i.e., construct a bipartite graph G with parts A and B such that $|A| = |B| = n$ so that $\delta(G) = \left\lfloor \frac{n-1}{2} \right\rfloor$ and G has no perfect matching.

Show that any red/blue coloring of the edges of a complete graph on n vertices contains a monochromatic tree on n vertices.

For an integer n, let K_n^L be the complete graphs on n vertices with an additional loop on each vertex, i.e., $K_n^L = (V, E)$ is a multigraph with loops such that $|V| = n$ and $E = \binom{V}{2} \cup V$. Decide whether the following Ramsey-type statement is true or false: for any integer k there exists an integer n such that any red/blue coloring of $E(K_n^L)$ contains a monochromatic copy of K_k^L.

For given integers k, ℓ and m, recall that $R(k, \ell, m)$ is the smallest integer N such that any red/blue/green coloring of $E(K_N)$ contains at least one of the following subgraphs: a red copy of K_k, a blue copy of K_ℓ, or a green copy of K_m.
Prove that

$$R(k, \ell, m) \leq \frac{(k + \ell + m - 3)!}{(k-1)!(\ell-1)!(m-1)!}.$$

An $n \times n$ *Latin square* is a table with n rows and n columns, where each cell contains one number between 1 and n in such a way that in each row every number appears exactly once, and also in every column each number appears exactly once (similarly as in Sudoku). See examples of a 3×3 and a 4×4 Latin squares.

1	2	3
2	3	1
3	1	2

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

Analogously, for $m \leq n$, an $m \times n$ *Latin rectangle* is a table with m rows and n columns where each cell contains one number between 1 and n in such a way that in each row every number appears exactly once, and in every column each number appears at most once.
Prove that for any $m \times n$ Latin rectangle there exists an $n \times n$ Latin square so that the first m lines of the Latin square are equal to the lines of the rectangle.

Recall that Ford-Fulkerson Theorem states that in every network the value of a maximum $s \to t$-flow is equal to the capacity of a minimum s,t-cut. Use this to establish an alternative (and short) proof of Hall's Theorem.

Let $G = (V, E)$ be a simple graph such that all the vertices except a one have degree at most 3, i.e., there is a vertex $x \in V$ so that $\deg_G(u) \leq 3$ for all $u \in V \setminus \{x\}$. Show that G is 4-colorable.

Let $G = (V, E)$ be a simple graph. Show that there exists an ordering of V such that the greedy coloring algorithm will find a coloring of G with $\chi(G)$ colors.

Recall an *orientation* of a simple graph $G = (V, E)$ is a function $o : E \to V$ that each $e \in E$ assigns one of its endpoints, which is then called the *head* of e. The other endpoint of e is called the *tail* of e. A triple (V, E, o), where (V, E) is a simple graph and o an orientation of E is called an *oriented graph*. An *oriented path / oriented cycle* in an oriented graph is a path / cycle where each edge is traversed from its tail to its head. An orientation of a simple graph G is called *acyclic* if the resulting oriented graph contains no oriented cycle.
Show that every simple graph G has an acyclic orientation.

Prove that a simple graph $G = (V, E)$ is k-colorable if and only if there exists an acyclic orientation o of its edges so that (V, E, o) contains no oriented path of length k.

Let $G = (V, E)$ be a simple graph with $\chi(G) = k$. Show that $|E| \geq \binom{k}{2}$.

Let $G = (V, E)$ be a simple graph with $\chi(G) = k$. Show that G contains a subgraph H such that $\chi(H) = k$ and for every vertex $v \in V(H)$ we have $\deg_H(v) \geq k - 1$.

Let $G = (V, E)$ be a simple graph on n vertices. Suppose $V = \{v_1, \ldots, v_n\}$ and consider the following graph G' on $2|V| + 1$ vertices $v'_1, v'_2, \ldots, v'_{2n}, z$:

- On the first $|V|$ vertices, put a copy of G,
- For any $i \in \{n+1, n+2, \ldots, 2n\}$ and $j \in \{1, 2, \ldots, n\}$, connect v'_i to v'_j if and only if v_{i-n} is adjacent to v_j in G,
- the vertices $\{v'_{n+1}, v'_{n+2}, \ldots, v'_{2n}\}$ form an independent set in G', and
- the vertex z is adjacent to all $v'_{n+1}, v'_{n+2}, \ldots, v'_{2n}$.

For example, if G is an edge, then G' is a 5-cycle.

a) Prove that if G is triangle-free, then G' is triangle-free as well.

b) Show that $\chi(G') = \chi(G) + 1$.

Let $G = (V, E)$ be a loopless multigraph and let $H \subseteq G$ be a subgraph of G with k vertices such that k is odd and $k \geq 3$. Show that

$$\chi'(G) \geq \left\lceil \frac{2|E(H)|}{k-1} \right\rceil .$$

Let $G = (V, E)$ be a loopless multigraph with maximum degree $\Delta(G) = \Delta$. In the lecture, we have shown that $\chi'(G) \leq 3\lceil \frac{\Delta}{2} \rceil$. Now, prove that

$$\chi'(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor \quad \text{(which is the same for } \Delta \text{ even, but better for } \Delta \text{ odd).}$$

(Hint: distinguish the cases $\mu(G) \leq \Delta/2$ and $\mu(G) > \Delta/2$. In the first case, use Vizing's theorem; otherwise do something and then use the bound from the lecture.)

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a) Construct a 5-regular simple graph that is planar.

b) What is the minimum number k of vertices of such a graph?

c) If your graph in (a) had more vertices than k, construct another 5-regular planar graph that has exactly k vertices.

Find a planar graph that is isomorphic to its dual.

Let H be a simple graph with maximum degree at most 3. Show that a simple graph G contains a subdivision of H if and olny if G contains H as a minor.

Let G be a simple graph that contains K_5 as a minor. Prove that G contains a subdivision of K_5 or a subdivision of $K_{3,3}$.