MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #1: Paths, Cycles and Trees

Due Wednesday, October 5th, 2016, 14:30

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1. For each of the following statements decide whether it is true or false, and either prove it, or give a counterexample.

a) Let G be a graph on $n \geq 2$ vertices with the vertex set $V = \{v_1, \ldots, v_n\}$. There exists two distinct vertices v_i and v_j such that $\deg(v_i) = \deg(v_j)$.

b) Let G be a graph and u, v, w be three vertices of G. If there is a cycle in G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

c) Let G be a graph and e, f, g be three edges of G. If there is a cycle in G containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

d) Let T be a tree on n vertices and let $v \in V(T)$ be a vertex of degree k. Then T contains at least k leaves, i.e., vertices of degree 1.

2. Let $G = (V, E)$ be a graph, and let $\overline{G}$ be the complement of G, i.e., the graph $(V, \overline{E})$, where $\overline{E} := \binom{V}{2} \setminus E$. Show that if G is not connected, then $\overline{G}$ is connected.

3. Let G be a graph with at least one vertex such that for every pair of vertices $u, v \in V(G)$, there is a path in G from u to v of length at most k. Show that if G is not a tree, then it contains a cycle of length at most $2k + 1$.

4. Let G be a connected graph which contains no path with length larger than k. Show that every two paths in G of length k have at least one vertex in common.

5. Let T be a tree, and let $T_1, T_2, \ldots, T_k$ be connected subgraphs of T so that $V(T_i) \cap V(T_j) \neq \emptyset$ for all i, j with $1 \leq i < j \leq k$. Show that

$$\bigcap_{i=1}^{k} V(T_i) \neq \emptyset.$$

[*Hint: Delete a leaf and use induction on* $|V(T)|$.]