*Instructions:* The exam is 3 hours long and contains 6 questions. The total number of points is 100. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. **Justify all your answers!**

**Q1** Let $G$ be the graph depicted in Figure 1.

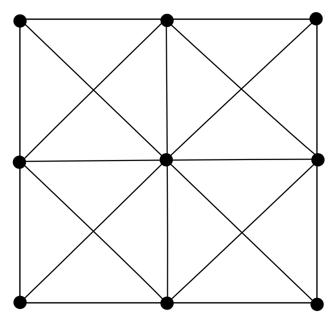
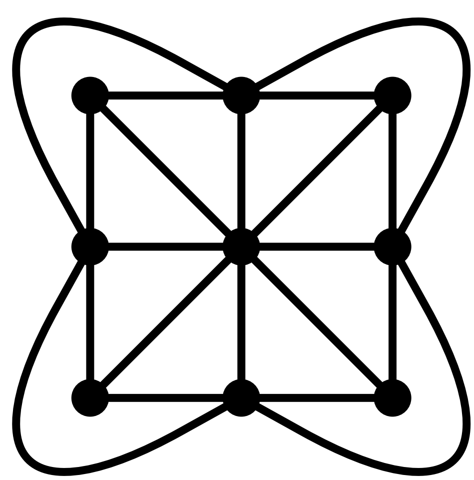


Figure 1: The graph in the question Q1.

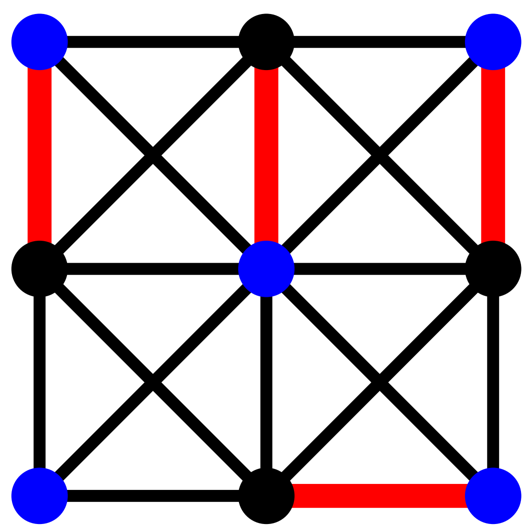
**a)** Is $G$ planar? *(4 points)*

**Solution:** Yes. See the following drawing of $G$.



**b)** Find $\nu(G)$ and $\tau(G)$. *(4 points)*

**Solution:** $G$ has 9 vertices, so $\nu(G) \leq \lfloor \frac{9}{2} \rfloor = 4$. On the other hand, the red edges on the picture below shows a matching of size 4 in $G$. We already know that $\tau(G) \geq \nu(G) = 4$. The blue vertices on the picture below shows a vertex cover of size 5; in the rest we show that $\tau(G) > 4$. Suppose there is a vertex cover $X$ in $G$ of size 4. Then $X$ must contain the middle vertex $v$ of $G$, otherwise $X$ needs to contain all the other eight vertices. However, that implies that the set $X - v$ is a vertex cover of size 3 in the graph $G - v \cong C_8$. But that is a contradiction (for example recall that $\alpha(C_8) = 4$ and $\alpha(H) + \tau(H) = |V(H)|$ for any graph $H$).



**c)** Find $\chi(G)$. *(4 points)*

**Solution:** Since $G$ contains $K_4$ as a subgraph $\implies \chi(G) \geq 4$. See on the left picture below a 4-coloring of $G$ which shows $\chi(G) \leq 4$.

**d)** Find $\chi'(G)$. *(4 points)*

**Solution:** The middle vetex has degree 8 $\implies \chi'(G) \geq 8$. See on the right picture below an 8-edge-coloring of $G$ which shows $\chi'(G) \leq 8$.

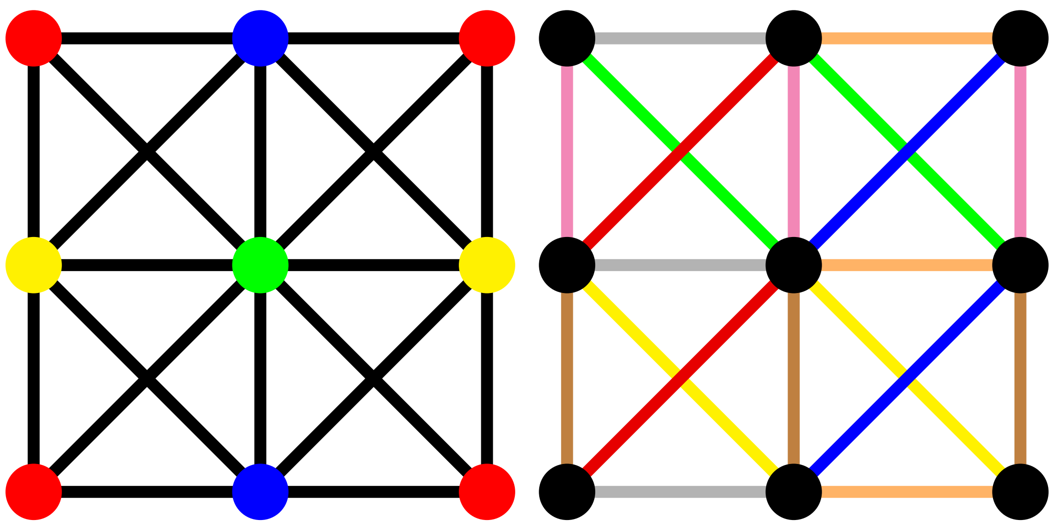


Figure 2: A 4-coloring of $G$ from Q1 with red,blue,green and yellow. An 8-edge-coloring of $G$ from Q1 with red,blue,green,yellow,orange,gray,purple and brown.

**Q2** Let $\overrightarrow{G} = (V, E)$ be the oriented graph with the two specific vertices $s$ and $t$ and with the capicities $c : E \to \mathbb{Z}_+$ depicted in Figure 3.

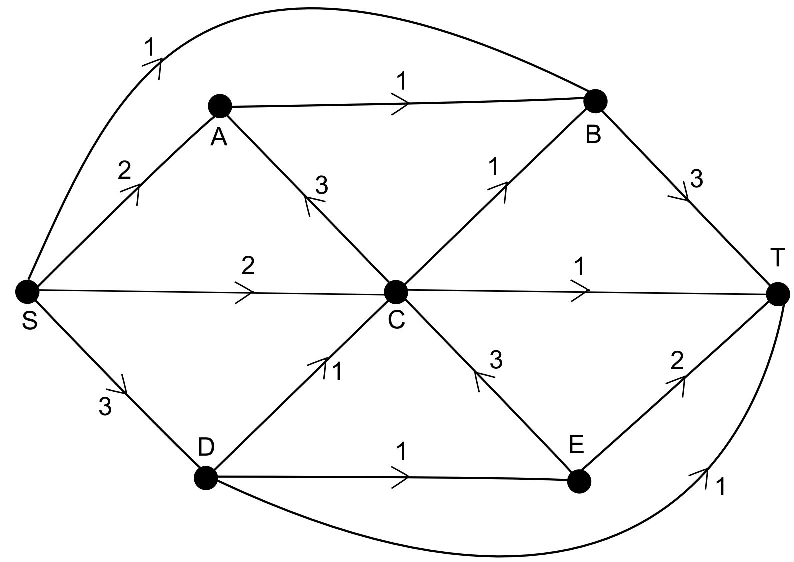
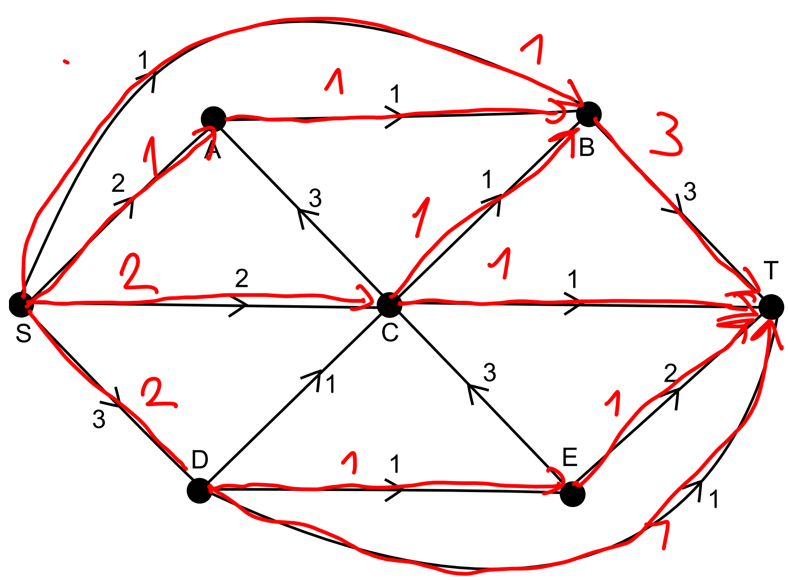



Figure 3: The oriented graph in the questions Q2 and Q3.

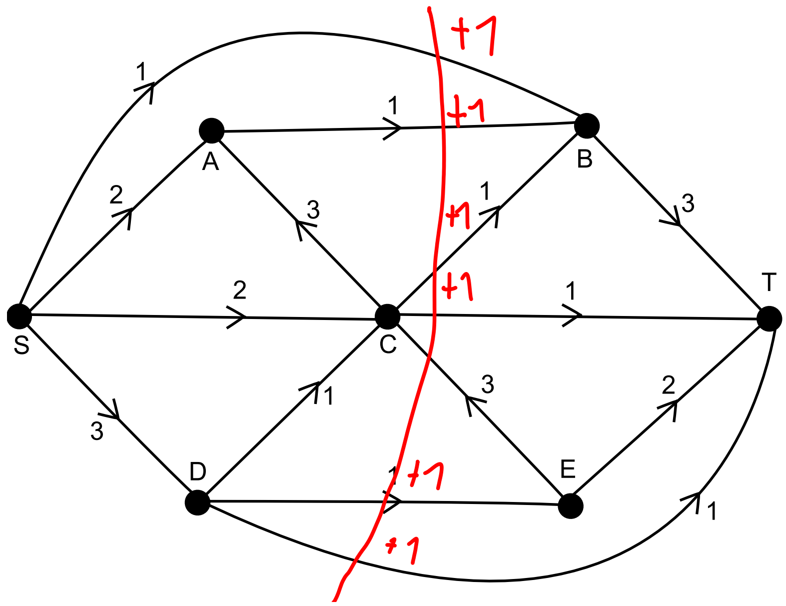
**a)** Find a maximum flow from the vertex $s$ to the vertex $t$. *(8 points)*

**Solution:** See the following flow from $s$ to $t$ of value 6.



**b)** Find a minimum $s,t$-cut. *(8 points)*

**Solution:** See the following $s,t$-cut of capacity 6.

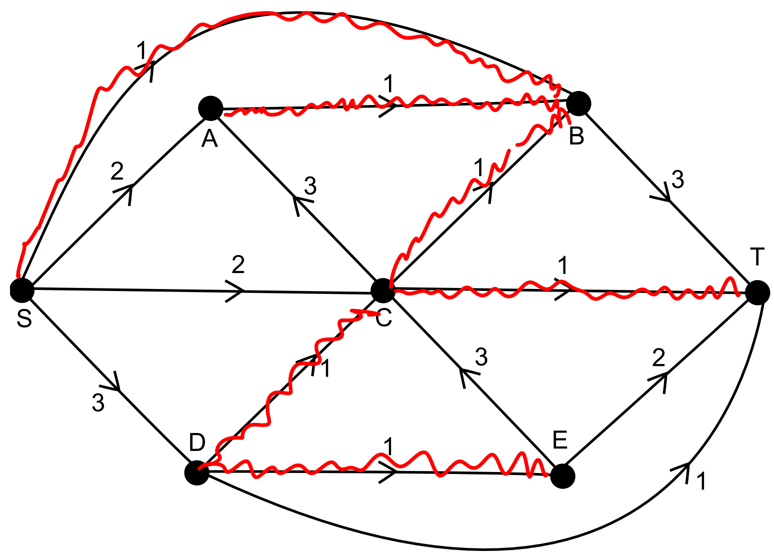



The construction of a flow from $s$ to $t$ of value 6 in (a) and the construction of an $s,t$-cut with capacity 6 in (b) yields that the flow is a maximum flow and the $s,t$-cut is a minimum cut.

**Q3** Let $G = (V, E)$ be the simple graph with weights $w : E \to \mathbb{Z}_+$ obtained from the oriented graph depicted in Figure 3 by replacing each oriented edge by a non-oriented one that has the same weight.

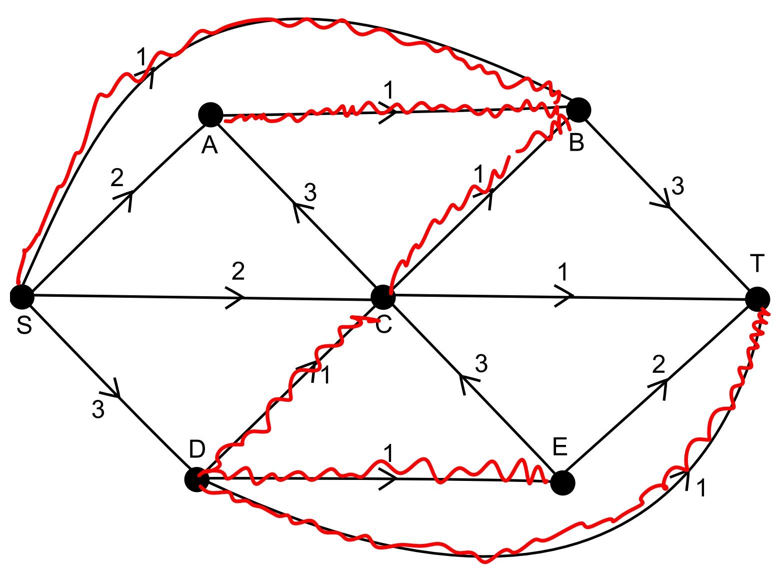
**a)** Find a minimum-cost spanning tree in $G$. *(8 points)*

**Solution:** See a minimum spanning tree of the total weight 6:




**b)** Does $G$ have a unique minimum-cost spanning tree. *(8 points)*

**Solution:** No! See another spanning tree of the total weight 6:



**Q4** Let $k \geq 1$ be an integer, and let $G$ be a connected $2k$-regular graph. Show that $G$ is 2-edge-connected. *(17 points)*

**Solution:** Suppose for contradiction that $G$ contains a cut-edge $e = \{u, w\}$, and let $C_u$ be the connected components of $G - e$ that contains the vertex $u$. It follows that the vertex $u$ in the component $C_u$ has degree $2k - 1$ and all the other vertices have still the degree $2k$. But this is impossible, because the hand-shaking lemma says that in any simple graph the number of the odd-degree vertices must be even.

**Q5** Let $G$ be a simple planar graph. Prove that if $G$ contains no cycle of length five or less, then $\chi(G) \leq 3$. *(17 points)*

**Solution:** We prove the statement by induction on $|V(G)|$. If $|V(G)| \leq 3$, then the statement is indeed trivially true. In the rest, we assume $|V(G)| \geq 4$.

If $G$ is disconnected, we can use the induction hypothesis on each of its components to find a 3-coloring of each component. The union of these 3-colorings yields a 3-coloring of the whole graph $G$.

Now suppose $G$ is connected and has a cut-vertex $x$. Let $C$ be one of the connected components of $G-x$ and let $V_1 := V(C) \cup \{x\}$ and $V_2 := V(G) \setminus V(C)$. It follows from the induction hypothesis that the two subgraphs $G_1$ and $G_2$ induced by $V_1$ and $V_2$, respectively, are both 3-colorable. Moreover, we can find a 3-coloring $c_1$ of $G_1$ and a 3-coloring $c_2$ of $G_2$ such that the vertex $x$ is colored both in $c_1$ and $c_2$ with the color 1. But then the union of these two 3-colorings again yields a 3-coloring of the whole graph $G$.

For the rest of the proof, suppose that $G$ is 2-connected and consider any drawing of $G$. Since $G$ is 2-connected, the boundary of any region $R$ in the drawing consists of a cycle in $G$. But $G$ has no cycles of length at most 5, so $\ell(R) \geq 6$. Therefore,

$$2|E(G)| = \sum_{R \text{ region}} \ell(R) \geq 6 \cdot Reg(G).$$

This yields that $Reg(G) \leq |E(G)|/3$ and from the Euler's formula we conclude

$$2 = |V(G)| + Reg(G) - |E(G)| \leq |V(G)| - 2|E(G)|/3.$$

A basic algebra manipulation with the last inequality yields that

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| \leq 3|V(G)| - 6,$$

which in-turn implies that $G$ must contain a vertex $x$ of degree 2. By the induction hypothesis applied on $G - x$, we can find a 3-coloring of $G - x$, which is then easily extended to $G$ by coloring $x$ with any color that is not present on its two neighbors.

**Q6** Let $K_4^-$ be the 4-vertex graph obtained from $K_4$ by removing one edge. How many non-isomorphic simple 2-connected graphs $G = (V, E)$ are there with $|V| = 1000$ such that $G$ has no $K_4^-$-minor? *(18 points)*

**Solution:** The answer is 1 and the only graph $G$ is $C_{1000}$. If $G$ is 2-connected, then it must contain a cycle. Let $C$ be the longest cycle in $G$. If $G$ is not isomorphic to $C_{1000}$, then either $C$ is a Hamilton cycle and $G$ contains at least one chord of $C$, or, the length of $C$ is at most 999. If $C$ is a Hamilton cycle and $G$ contains at least one chord of $C$, then clearly $C$ and this chord is a minor of $K_4^-$.

On the other hand, if $C$ does not cover all the vertices, then let $x \in V(G) \setminus V(C)$. By the 2-connectivity assumption on $G$, it contains two paths $P_1$ and $P_2$ so that $V(P_1) \cap V(P_2) = \{x\}$, $V(C) \cap V(P_1) = \{w_1\}$, $V(C) \cap V(P_2) = \{w_2\}$, the endpoints of $P_1$ are $x$ and $w_1$, and the endpoints of $P_2$ are $x$ and $w_2$. But then $C \cup P_1 \cup P_2$ is a minor of $K_4^-$.