1. Let $\mathcal{G}$ be the set of all simple graphs G with $|V(G)| = 9$ such that

- G has three vertices of degree 3,
- G has three vertices of degree 5, and
- G has three vertices of degree 6.

a) Construct a graph $G \in \mathcal{G}$ that has a Hamilton cycle.

Solution:

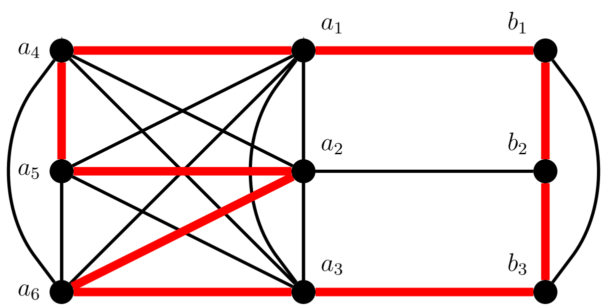

Figure 1: A graph $G \in \mathcal{G}$ with a Hamilton cycle used in the solution of Problem 1a).

Consider the 9 vertex graph which consists of six vertices $\{a_1, \dots, a_6\}$ that induces a copy of K_6 and three vertices $\{b_1, b_2, b_3\}$ that induces a copy of K_3, plus three extra edges $\{a_1, b_1\}$, $\{a_2, b_2\}$ and $\{a_3, b_3\}$. It follows that the vertices a_1, a_2 and a_3 have degree 6, the vertices a_4, a_5 and a_6 have degree 5, and the vertices b_1, b_2 and b_3 have degree 3. It is also easy to find a Hamilton cycle in G; see Figure 1.

b) Prove that every graph $G \in \mathcal{G}$ is 2-connected.

Solution: Suppose there is $G \in \mathcal{G}$ that is not 2-connected. By Menger's theorem, there exists $S \subseteq V(G)$ such that $|S| \leq 1$ and $G - S$ is disconnected. First let us look at the case $|S| = 1$.

Suppose there is a vertex $v \in V(G)$ such that $G' := G - v$ is not connected. Let C_1 be a connected component of G' that contains at least one of the vertices that has degree 6 in G (there are at least two such vertices; note that the vertex v might had degree 6), and let C_2 be a connected component of G' different from C_1. It follows that C_1 contains a vertex of degree at least 5, so $|C_1| \geq 6$. Therefore, $|C_2| \leq 2$. But this means that each vertex from C_2 can have at most 2 neighbors in G, which contradicts $G \in \mathcal{G}$.

It remains to argue that there is no cut of size zero, i.e., the graph G itself is connected. Indeed, otherwise let C_1 and C_2 be two connected components in G. Since every vertex in G has degree at least 3, it follows that $|C_1| \geq 3$. Fix $v \in C_1$ arbitrarily and consider $G' := G - v$. It follows that $C_1 - v$ and C_2 are two connected components of G', so G' is disconnected, which contradicts the conclusion of the previous paragraph.

c) Is there a graph $G \in \mathcal{G}$ that is bipartite?
(*If yes, construct a one. If no, prove it!*)

Solution: NO. Suppose there is $G \in \mathcal{G}$ such that G is bipartite, and let A and B be the parts of a bipartition of G. Without loss of generality, the part A contains a vertex of degree 6. Therefore, $|B| \geq 6$ and hence $|A| \leq 3$. So every vertex of G that has degree at least 4 must be contained in A, but there are six such vertices in G; a contradiction.

d) Is it true that every graph $G \in \mathcal{G}$ has a Hamilton cycle?
(*If yes, prove it! If no, construct a graph $G \in \mathcal{G}$ and show G has no Hamilton cycle.*)

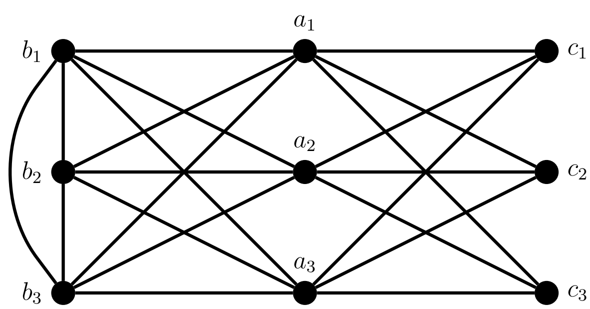

Figure 2: A graph $G \in \mathcal{G}$ with no Hamilton cycle used in the solution of Problem 1d).

Solution: NO. Let G be the graph depicted in Figure 2. It is straightforward to verify that $G \in \mathcal{G}$. Now suppose that G contains a Hamilton cycle C. Since C is a cylce, the number of edges of C incident to each vertex c_i, where $i \in \{1, 2, 3\}$, is equal to two. All these six edges must have their other endpoints in $\{a_1, a_2, a_3\}$. However, there also must be an edge in C with one endpoint in $\{b_1, b_2, b_3\}$ and the other endpoint in $\{a_1, a_2, a_3\}$ (in fact, there must be at least two such edges). But this means that the vertices a_1, a_2 and a_3 are in C incident to at least 7 edges contradicting that C is a cycle.

2. Let $G = (V, E)$ be a simple graph. A set $A \subseteq V$ is called *an independent set in* G if the induced subgraph $G[A]$ contains no edge, i.e., if every edge $e \in E$ is incident to at most one vertex in A. Define $\alpha(G)$ to be the maximum cardinality of an independent set in G.
Suppose $G = (V, E)$ is a simple graph. Prove that there is an integer $\ell \leq \alpha(G)$ and a collection of paths $P_1, P_2, \ldots, P_\ell$ such that for every vertex $v \in V$ there exists *exactly one* path P_i, where $i \in \{1, \ldots, \ell\}$, with $v \in V(P_i)$.

Solution: First, let us show that for some integer ℓ (maybe $\ell > \alpha(G)$), there exists a collection of paths $P_1, \ldots, P_\ell$ with the property that every vertex $v \in V$ is in exactly one P_i. Indeed, for $\ell = |V|$ the task is trivial – just consider $|V|$ one-vertex paths, each containing a different vertex of G.
Now let ℓ_0 be the minimum ℓ such that there exists a collection of paths $P_1, \ldots, P_\ell$ with the property that every vertex $v \in V$ is in exactly one P_i. Suppose $\ell_0 > \alpha(G)$, and let $P_1, \ldots, P_{\ell_0}$ be the appropriate paths. For each $i \in \{1, \ldots, \ell_0\}$, let u_i be one of the two end-vertices of the path P_i. Set $U := \{u_1, u_2, \ldots, u_{\ell_0}\}$. Since the paths $P_1, \ldots, P_{\ell_0}$ were disjoint, we have $|U| = \ell_0 > \alpha(G)$. But this means U cannot be independent, so it contains an edge $\{u_i, u_j\}$ for some distinct $i, j \in \{1, \ldots, \ell_0\}$. Let Q be the graph with the vertex set $V(P_i) \cup V(P_j)$ and the edge set $E(P_i) \cup E(P_j) \cup \{u_i, u_j\}$. It follows that Q is a path. But then replacing the paths P_i and P_j with Q yields a collection of $\ell_0 - 1$ paths such that every vertex $v \in V$ is in exactly one of them; a contradiction with the minimality of ℓ_0.

3. Let $G = (V, E)$ be a bipartite graph with parts A and B. Recall that for $S \subseteq A$,

$$N_G(S) := \{x \in V | \{s, x\} \in E \text{ for some } s \in S\} .$$

a) Show that if M is a matching in G and $S \subseteq A$, then $|A| - |M| \geq |S| - |N_G(S)|$.

Solution:

Let $U_A \subseteq A$ be the set of vertices from A that are "unmatched" by M, i.e., not incident to any edge $e \in M$. It follows that $|U_A| = |A| - |M|$. Now let $U_S := U_A \cap S$ be the set of vertices from S that are unmatched by M. Clearly $|U_A| \geq |U_S|$, so it is enough to show $|U_S| \geq |S| - |N_G(S)|$.

Let m_S be the number of vertices from S that are matched by M. On one hand, $m_S = |S| - |U_S|$. On the other hand, $m_S \leq |N_G(S)|$. Therefore, $|U_S| \geq |S| - |N_G(S)|$ and hence also $|A| - |M| \geq |S| - |N_G(S)|$.

b) Show that M is a matching in G of the maximum size if and only if there exists some $S_0 \subseteq A$ such that $|A| - |M| = |S_0| - |N_G(S_0)|$.

Solution: First we show that if M is a maximum matching, then we can find a set S_0 such that $|A| - |M| = |S_0| - |N_G(S_0)|$. Let X be a minimum vertex cover of G. Since G is bipartite, by König's theorem we know $|M| = |X|$ so it is enough to find S_0 such that $|A| - |X| = |S_0| - |N_G(S_0)|$.

Let $X_A := X \cap A$, $X_B := X \cap B$ and $S_0 := A \setminus X_A$. By definition, $|S_0| = |A| - |X_A|$. Also, since X is a vertex cover of G, it holds that $N_G(S_0) \subseteq X_B$. On the other hand, if there would be a vertex $u \in X_B \setminus N_G(S_0)$, then $X - u$ is also a vertex cover of G contradicting the minimality of X. We conclude that $N_G(S_0) = X_B$. Therefore,

$$|A| - |M| = |A| - |X| = |A| - |X_A| - |X_B| = |S_0| - |N_G(S_0)|.$$

It remains to show that if $S_0 \subseteq A$ is such that $|A| - |M| = |S_0| - |N_G(S_0)|$ for some matching M, then M must be a matching of maximum size. Indeed, if there exists a matching M' such that $|M| < |M'|$, then

$$|A| - |M'| < |A| - |M| = |S_0| - |N_G(S_0)|.$$

But this is a contradiction with the part (a).