1. For each of the following statements decide whether it is true or false, and either prove it, or give a counterexample.

a) Let $G$ be a graph on $n \geq 2$ vertices with the vertex set $V = \{v_1, \ldots, v_n\}$. There exists two distinct vertices $v_i$ and $v_j$ such that $\deg(v_i) = \deg(v_j)$.

Solution: The question was stated ambiguously, since the answer depends on whether we consider only simple graphs or not. Both answers will be accepted, if they were correctly argued.

If we assume that $G$ is simple, then the statement is True. Every vertex in $G$ has the degree between 0 and $n-1$, and there are $n$ vertices in total. If all the degrees would be different, then $G$ must contain a vertex $u$ with $\deg(u) = 0$, and a vertex $v$ with $\deg(v) = n-1$. However, that means that $u$ is an isolated vertex (in particular, $u$ is not adjacent to $v$), and $v$ is a vertex adjacent to all the $n-1$ vertices different from $v$ (in particular, $v$ is adjacent to $u$); a contradiction.

If $G$ does not have to be simple, so in particular, multiple edges are allowed, the statement is False, as can be seen in Figure 1.

![Figure 1: A counterexample for Problem 1a) in the case $G$ does not have to be a simple graph.](image1)

b) Let $G$ be a graph and $u, v, w$ be three vertices of $G$. If there is a cycle in $G$ containing $u$ and $v$, and a cycle containing $v$ and $w$, then there is a cycle containing $u$ and $w$.

Solution: False. See Figure 2.

c) Let $G$ be a graph and $e, f, g$ be three edges of $G$. If there is a cycle in $G$ containing $e$ and $f$, and a cycle containing $f$ and $g$, then there is a cycle containing $e$ and $g$.

Solution: True. Let $e = \{u_1,v_1\}$ and $f = \{u_2,v_2\}$. Fix an arbitrary cycle $C_1$ containing $e$ and $f$. Without loss of generality,

$$C_1 = u_2, \ldots, u_1, e, v_1, \ldots, v_2, f, u_2$$

where $P$ and $Q$ are the paths on $C_1$ between $u_1$ and $u_2$, and $v_1$ and $v_2$, respectively, that both avoid the edges $e$ and $f$. Clearly, $P$ and $Q$ are vertex-disjoint.
Now, let $C_2$ be a cycle containing $f$ and $g$. We may assume $C_2$ does not contain the edge $e$, otherwise there is nothing to prove. Define $p$ to be the first vertex on $P$ starting from $u_1$ that is contained in $C_2$. Such a vertex must exist, since $u_2$ is a vertex of $C_2$ (note it might be that $p = u_1$). Analogously, let $q$ be the first vertex on $Q$ starting from $v_1$ that is contained in $C_2$ (again, it might be that $q = v_1$). Let $R_1$ be the path on $C_1$ between $p$ and $q$ that contains the edge $e$. It follows from the construction that $V(R_1) \cap V(C_2) = \{p, q\}$. Now set $R_2$ to be the path on $C_2$ between $p$ and $q$ that contains the edge $g$. The union of the edges of $R_1$ and $R_2$ forms a cycle that contains both $e$ and $g$.

d) Let $T$ be a tree on $n$ vertices and let $v \in V(T)$ be a vertex of degree $k$. Then $T$ contains at least $k$ leaves, i.e., vertices of degree 1.

Solution: True. Let $L$ be the set of leaves in $T$. Since $T$ is a tree, $|V(T)| = |E(T)| + 1$. On the other hand,

$$2|V(T)| - 2 = 2|E(T)| = \sum_{u \in V(T)} \deg(u) = |L| + k + \sum_{u \in V(T) \setminus L} \deg(u).$$

Since every vertex $u \in V \setminus L$ has degree at least 2, it follows that

$$\sum_{\substack{u \in V(T) \setminus L \atop u \neq v}} \deg(u) \geq 2(|V(T)| - |L| - 1) = 2|V(T)| - 2|L| - 2.$$

Combining the two derivations together, we conclude that

$$2|V(T)| - 2 \geq |L| + k + 2|V(T)| - 2|L| - 2 = k - |L| + 2|V(T)| - 2,$$

which after rearranging the terms yields $|L| \geq k$.

2. Let $G = (V, E)$ be a graph, and let $\overline{G}$ be the complement of $G$, i.e., the graph $(V, \overline{E})$, where $\overline{E} := \binom{V}{2} \setminus E$. Show that if $G$ is not connected, then $\overline{G}$ is connected.

Solution: Let $C$ be an arbitrary connected component of $G$, and let $D := V(G) \setminus C$. Since $G$ is not connected, $D \neq \emptyset$. Fix two vertices $u \in C$ and $v \in D$. It follows that in the graph $\overline{G}$, any vertex
in $C$ is connected to any vertex in $D$ by an edge. Moreover, for any two vertices $c_1, c_2 \in C$, there is a path of length two in $\overline{G}$ between $c_1$ and $c_2$ via $v$. Analogously, for any two vertices $d_1, d_2 \in D$, there is a path of length two between $d_1$ and $d_2$ via $u$. So $\overline{G}$ is connected.

3. Let $G$ be a graph with $|V(G)| \geq 1$ where for every pair of vertices $u, v \in V(G)$, there is a path in $G$ from $u$ to $v$ of length at most $k$. Show that if $G$ is not a tree, then it contains a cycle of length at most $2k + 1$.

Solution: Clearly, $G$ is connected and contains a cycle. Let $C$ be a cycle in $G$ of the smallest length and let $v_1, v_2, \ldots, v_k$ be the vertices of $C$ in order. Suppose for a contradiction that $\ell \geq 2k + 2$. Let $P$ be the shortest path from $v_1$ to $v_{k+2}$ in $G$. Then $P$ has length at most $k$ and it follows that $P \subseteq C$. Thus there exists a subpath $Q$ of $P$ with distinct ends $v_i, v_j \in V(P)$ and otherwise disjoint from $C$. The union of $Q$ with each of the two paths in $C$ with ends $v_i$ and $v_j$ is a cycle, and so each of these cycles must have length at least $\ell$. The sum of their lengths, however, is equal to $\ell + 2|E(Q)| \leq \ell + 2|E(P)| \leq \ell + 2k < 2\ell$, a contradiction.

4. Let $G$ be a connected graph which contains no path with length larger than $k$. Show that every two paths in $G$ of length $k$ have at least one vertex in common.

Solution: Suppose for a contradiction that $P_1$ and $P_2$ are two vertex-disjoint paths of length $k$. Let vertices of $P_1$, where $1 \leq i \leq 2$, be $v_1^i, v_2^i, \ldots, v_{k+1}^i$, in order. Let $Q$ be a path with one end in $V(P_1)$ and another in $V(P_2)$ chosen to be as short as possible. Let $v_1^m$ and $v_2^n$ be the ends of $Q$, where $1 \leq m, n \leq k + 1$. We can suppose without loss of generality that $m, n \geq \lceil k/2 + 1 \rceil$. Then a path obtained by taking the union of the subpath of $P_1$ from $v_1^1$ to $v_1^n$, the path $Q$ and the subpath of $P_2$ from $v_2^n$ to $v_2^1$ has at least $m + n \geq k + 2$ vertices, a contradiction.

5. Let $T$ be a tree, and let $T_1, \ldots, T_k$ be connected subgraphs of $T$ so that $V(T_i \cap T_j) \neq \emptyset$ for all $i, j$ with $1 \leq i < j \leq k$. Show that

$$\bigcap_{i=1}^{k} V(T_i) \neq \emptyset.$$

Solution: Proof by induction on $|V(T)|$. The base case $|V(T)| = 1$ is trivial. For the induction step, let $v$ be a leaf of $T$ and let $u$ be the unique vertex of $T$ adjacent to $v$. Let $T' = T \setminus v$ and let $T'_i = T_i \setminus v$ for $i = 1, 2, \ldots, k$. If $V(T'_i \cap T'_j) \neq \emptyset$ for all $i, j$ with $1 \leq i < j \leq k$, then we can apply the induction hypothesis to $T'$ to complete the proof. Thus we may assume, without loss of generality, that $V(T'_1) \cap V(T'_2) = \emptyset$. It follows that $V(T_1) \cap V(T_2) = \{v\}$. Thus either $u \notin V(T_1)$ or $u \notin V(T_2)$. Without loss of generality, we have $V(T_1) = \{v\}$. Therefore $v \in V(T_i)$ for every $1 \leq i \leq k$ by the assumption and $v \in V(T_1 \cap T_2 \cap \ldots \cap T_k)$, as desired.