1. Let $G = (V, E)$ be a simple graph and let $U \subseteq V$. We define $G \oplus_U \{v\}$ to be the graph obtained from $G$ by adding a new vertex $v$, which is then joined to every vertex in $U$. In other words, $G \oplus_U \{v\} = (V \cup \{v\}, E \cup \{(u, v) : u \in U\})$.

a) Prove that if $G = (V, E)$ is a $k$-connected simple graph and $U \subseteq V$ has size $k$, then the graph $G \oplus_U \{v\}$ is $k$-connected as well.

**Solution:** Suppose for a contradiction $G' := G \oplus_U \{v\}$ is not $k$-connected. By Menger’s theorem, there exists a vertex cut $S \subseteq V(G')$ of size at most $k - 1$. Clearly, if $v \in S$, then $G' - S$ is actually a subgraph of $G$ with at least $|V| - k - 2$ vertices, which is definitely connected (in fact, it is even 2-connected) by the connectivity assumption on $G$.

Now consider $v \notin S$. Let $C_1$ and $C_2$ be different connected components of $G' - S$. We claim that both $C_1$ and $C_2$ contain a vertex from the set $V$. If not, then one of the components, say $C_1$, would contain only the vertex $v$. However, since $|U| = k$, there is at least one vertex $u \in U \setminus S$, and this vertex must be in $C_1$ as well; a contradiction.

Let $u_1 \in V(C_1) \cap V$ and $u_2 \in V(C_2) \cap V$. It follows that every path in $G$ between $u_1$ and $u_2$ have to pass through the set $S$, which is a contradiction with $G$ being $k$-connected.

b) For every integer $k > 1$, find a simple graph $G_k = (V_k, E_k)$ on at least $k + 1$ vertices and a vertex-subset $U \subseteq V_k$ of size $k$ such that $G_k$ is not $k$-connected, however, $G_k \oplus_U \{v\}$ is $k$-connected.

**Solution:** There was a typo in the original statement – one has to assume $k > 1$ since the statement is clearly false for $k = 1$. The points for this part will not be counted to the regular score. You get a bonus point if you have spotted the mistake and constructed a counter-example for the case $k = 1$. You get extra 2 points if you have constructed the graphs $G_k$ for any $k \geq 2$.

Fix an integer $k \geq 2$. Let $V := \{v_1, v_2, \ldots, v_{k+1}\}$ and let $G_k := \left(V, \left(\begin{array}{c} V \\ 2 \end{array}\right) \setminus \{k, k+1\}\right)$.

In other words, $G_k$ is obtained from a complete graph on $k + 1$ vertices by removing one edge. Clearly, this graph is not $k$-connected because the set $\{v_1, \ldots, v_{k-1}\}$ is a vertex cut in $G_k$ of size $k - 1$. Let $U := \{2, 3, \ldots, v_{k+1}\}$, and $G_k' := G_k \oplus_U \{v\}$. We claim $G_k'$ is $k$-connected.

Indeed, consider $S \subseteq V(G_k')$ a vertex cut in $G_k'$. By Menger’s theorem, it is enough to show $|S| \geq k$. First, observe that for any $i \in \{2, 3, \ldots, k-1\}$, the vertex $v_i$ is connected to every other vertex in $G_k'$. Therefore, any vertex cut in $G_k'$ must contain all the vertices from $\{v_2, v_3, \ldots, v_{k-1}\}$, so $|S| \geq k - 2$. But $G_k' - \{v_2, v_3, \ldots, v_{k-1}\}$, i.e., the subgraph of $G_k'$ induced by $\{v_1, v_k, v_{k+1}, v\}$, is isomorphic to $C_4$, so $|S| \geq k - 1$. However, if $|S| = k - 1$, then by the argument above $S$ contains exactly one vertex from $\{v_1, v_k, v_{k+1}, v\}$. Therefore $G_k' - S$ is isomorphic to a path of length two, a contradiction.
2. Let \( G = (V, E) \) be a \( k \)-connected simple graph and \( U, W \subseteq V \) two vertex-subsets, each of size \( k \). Prove that there exist \( k \) pairwise vertex-disjoint paths \( P_1, \ldots, P_k \) such that for every \( i \in \{1, \ldots, k\} \), the path \( P_i \) have one endpoint in \( U \) and the other endpoint in \( W \).

**Solution:** Let \( G' := (G \cup_U u) \cup_W w \). By the part (a) of the previous exercise, \( G' \) is \( k \)-connected. Therefore, \( G' \) contains \( k \) internally disjoint paths \( Q_1, \ldots, Q_k \) between \( u \) and \( w \). For every \( i \in \{1, \ldots, k\} \), let \( P_i := Q_i - u - w \). It follows that these are \( k \) vertex-disjoint paths in \( G \), each with exactly one end in \( U \) and the other in \( W \).

3. Let \( G = (V, E) \) be a 2-connected simple graph. Show that for any triple of distinct vertices \( u, v, w \in V \) there is a path in \( G \) from \( u \) to \( v \) passing through \( w \), i.e., \( w \) is one of the inner vertices of the path.

**Solution:** Let \( G' := G \oplus_U z \) for \( U := \{u, v\} \). Again, the first part of Exercise 1 yields that \( G' \) is 2-connected. Hence \( G' \) contains 2 internally vertex-disjoint paths \( Q_1 \) and \( Q_2 \) between \( z \) and \( w \). Taking their union and removing the vertex \( z \) yields the desired path between \( u \) and \( v \) that passes through \( w \).

4. Let \( G = (V, E) \) be a 2-connected simple graph and \( v \in V \) a vertex of \( G \). Prove that there exists a vertex \( u \in V \) such that \( \{u, v\} \in E \) and the graph \( G - u - v \) is connected.

**Solution:** Let \( U \) be the set of neighbors of \( v \) in \( G \). Let \( T \) be a connected subgraph of \( G - v \) with the minimum number of edges such that \( U \subseteq V(T) \). It is easy to see that \( T \) is a tree, and that every leaf of \( T \) is a neighbor of \( v \). Let \( u \) be a leaf of \( T \). Then \( T - u \) is connected. Suppose for a contradiction that \( G - u - v \) is not connected and consider a component \( C \) of \( G - u - v \) which does not contain \( T - u \). Thus \( C \) contains no neighbor of \( v \) and so it is a connected component of \( G - u \). It follows that \( G - u \) is not connected, contradicting 2-connectivity of \( G \).

5. Let \( G = (V, E) \) be a directed graph (digraph) and for each edge \( e \in E \), let \( \phi(e) \geq 0 \) be a non-negative integer. Show that if for every vertex \( v \)

\[
\sum_{e \in \partial^-(v)} \phi(e) = \sum_{e \in \partial^+(v)} \phi(e),
\]

then there is a collection of directed cycles \( C_1, \ldots, C_k \) (possibly with repetition) so that for every edge \( e \) of \( G \), it holds that

\[
\{i : 1 \leq i \leq k, \ e \in E(C_i)\} = \phi(e).
\]

**Solution:** Induction on \( S := \sum_{e \in E(G)} \phi(e) \). Base case: \( S = 0 \) is trivial. For the induction step, it suffices to find a directed cycle \( C \) in \( G \) so that \( \phi(e) \geq 1 \) for every edge \( e \in E(G) \), as one can then apply the induction hypothesis to

\[
\phi'(e) := \begin{cases} 
\phi(e), & \text{if } e \notin E(C) \\
\phi(e) - 1, & \text{if } e \in E(C)
\end{cases}
\]

Let \( e \) be an edge of \( G \) with \( \phi(e) \geq 1 \), a tail \( u \) and a head \( v \). Then \( \phi \) restricted to \( E(G) - e \) is a \( u-v \)-flow of value 1. By Lemma 11.3 from the lecture notes, there exists a directed path \( P \) in \( G - e \) so that \( \phi \) is positive on every edge of the path. The path \( P \) together with \( e \) forms the desired cycle.