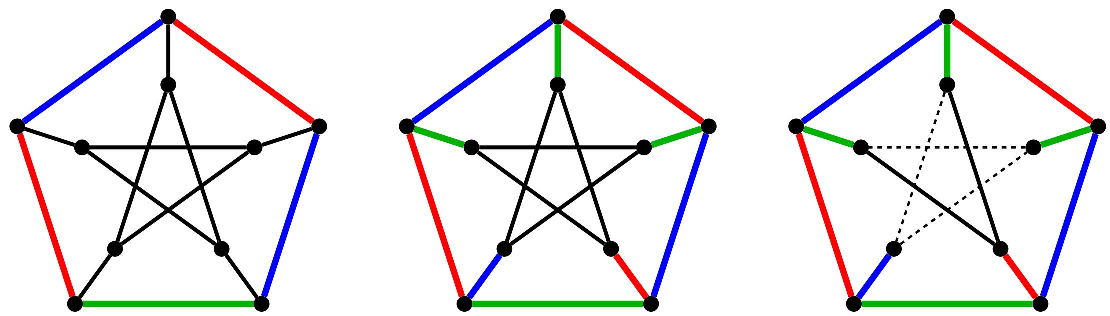
MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #5: Edge-colorings, Line graphs, Planar graphs

Due Wednesday, November 30th, 2016, 14:30

1a) Show that the Petersen graph has no 3-edge-coloring. *(2 points)*

Solution: Suppose for a contradiction the Petersen graph is 3-edge-colorable. Up to the permutation of the colors, the outer five-cycle must be colored as in the left-most picture. That also uniquely determines the colors of the other five edges incident to to the vertices of the outer cycle; see the middle picture. However, this now forces the two bold edges in the right-most picture to be both blue; a contradiction.

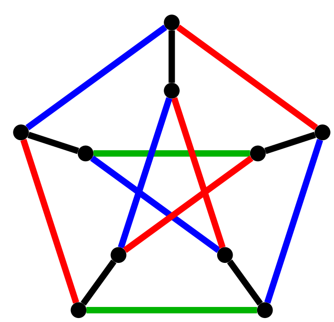


1b) Does the Petersen graph have a Hamilton cycle? *(1 point)*

Solution: No! Suppose there is a Hamilton cycle C in the Petersen graph, and let M be the edges of the Persen graph that are not contained in C. It follows that M is a perfect matching. On the other hand, the Petersen graph has 10 vertices, so we can split the edges of C into two perfect matchings N and O. It follows that M, N and O form a 3-edge-coloring of the Petersen graph, a contradiction with the part (a).

1c) Find a 4-edge-coloring of the Petersen graph. *(1 point)*

Solution: See the 4-edge-coloring below.



2. For $n \geq 2$, use the following steps to determine $\chi'(K_n)$ and construct an optimal edge-coloring.

a) For any odd integer $n \geq 3$, show that the complete graph K_n does not have an edge-coloring with $\Delta(K_n) = n - 1$ colors.

Solution: Indeed, K_n is an $(n-1)$-regular graph, so if it has an edge-coloring with $n-1$ colors, then each color class must form a perfect matching. But for n odd, K_n cannot have a perfect matching.

b) For any odd integer $n \geq 3$, prove that if c is an edge-coloring of K_n with n colors, then each color class of c contains $(n-1)/2$ edges.

(Note that $\chi'(K_n) \leq n$ by Vizing's Theorem.)

Solution: Consider an edge-coloring of K_n with n colors. Each color class is a matching, and since n is odd, any matching of K_n has size at most $(n-1)/2$ edges. However, each edge of K_n has one of the n colors and since

$$\binom{n}{2} = n \cdot \frac{n-1}{2},$$

we conclude that the bound $(n-1)/2$ on the size of a color class must be tight.

c) For any even integer $n \geq 2$, show that $\chi'(K_n) = n - 1$.

Solution: Consider any edge-coloring of K_{n-1} using $n-1$ colors. From the part (b), we know that each color class contains $(n-2)/2$ edges. In other words, for each color $i \in \{1, \dots, n-1\}$, there is exactly one vertex v_i that is not incident to any edge colored with i. Moreover, for different colors $i \neq j$, it holds that $v_i \neq v_j$. Adding a new vertex v_n and coloring the edge $\{v_i, v_n\}$ with the color i for all $i \in \{1, \dots, n-1\}$ yields an $(n-1)$-edge-coloring of K_n.

d) For any integer $n \geq 2$, explicitly construct an edge-coloring of K_n with $\chi'(K_n)$ colors.

Solution: As the hint suggested, we should show that for n being odd and $V(K_n) = \{0, \dots, n-1\}$, coloring the edge $\{i, j\}$ with $(i+j) \mod n$ yields an edge-coloring of K_n. Suppose for a contradiction that there are two edges $e_1 \neq e_2$ incident to some vertex i that are both colored with the same color, say $x \in \{0, \dots, n-1\}$. Let $e_1 = \{i, j\}$ and $e_2 = \{i, k\}$. Since $(i+j) \equiv x \equiv (i+k) \mod n$, we have $j \equiv k \mod n$. However, that means that $j = k$ contradicting $e_1 \neq e_2$.

If n is even, we let $n' := n - 1$ and $V(K_n) = \{0, \dots, n'-1, n'\}$. If $i, j \in \{0, \dots, n'-1\}$, we color the edge $\{i, j\}$ with $(i+j) \mod n'$, and the remaining edges $\{i, n'\}$, where $i \in \{0, \dots, n'-1\}$, we color with $(2i) \mod n'$. Since n' is odd, it follows that $2i \neq 2j \mod n'$ for any $i, j \in \{0, \dots, n'-1\}$ with $i \neq j$.

3. Let $G = (V, E)$ be a loopless multigraph. Recall that a *line graph* of G, which we denote by $L(G)$, is a simple graph H with the vertex set E, and two vertices e and f of H are adjacent if and only if the corresponding two edges in G are incident to the same vertex. In other words, $H = (E, F)$ where $F = \{\{e, f\} : e \cap f \neq \emptyset\}$.

a) Let $G = (V, E)$ be a loopless connected multigraph with an <u>even</u> number of edges, i.e., $|E|$ is even. Show that the graph $L(G)$ has a perfect matching.

Solution: Suppose for contradiction $L(G)$ does not have a perfect matching. By Tutte's theorem, there exists $S \subseteq E$ such that $k > |S|$ for $k := odd_{L(G)}(E \setminus S)$. It follows that the parity of k is the same as the parity of $|S|$, hence $k \geq |S| + 2$. Now look back to the graph G. The connected components of the subgraph of $L(G)$ induced by $E \setminus S$ are in one-to-one correspondence with the connected components of $G' := (V, E \setminus S)$. So G' has at least k connected components. However, each edge from S can connect at most two components of G' and since $|S| < k - 1$, G cannot be connected.

b) Let $G = (V, E)$ be a loopless connected multigraph with an <u>odd</u> number of edges. Show that $L(G)$ has a matching of size $\frac{|E|-1}{2}$.

Solution: Simply add an aribtrary edge to G connecting two different vertices and use the previous part. The perfect matching M in the line graph of the new graph contains a matching $M' \subseteq E$ of size $\frac{|E|-1}{2}$.

Alternatively, if G is not a tree, there is $e \in E$ such that $G' := G - e$ is connected. On the other hand, if G is a tree, then let v be a leaf and $G' := G - v$. In both cases, G' is connected $|E(G')|$ is even, and $L(G')$ is a subgraph of $L(G)$, so we use the part a).

4. Let $G = (V, E)$ be a planar graph drawn in the plane. Suppose that there exists a vertex v so that v belongs to the boundary of every region. Show that

$$\alpha(G) \geq \frac{|V| - 1}{2}.$$

Solution: Let $G' := G - v$. By the assumption, we have $Reg(G') = 1$. So G' is a forest and hence two-colorable. The larger color class has size $k \geq |V(G')|/2$ hence

$$\alpha(G) \geq \alpha(G') \geq \frac{|V(G')|}{2} = \frac{|V| - 1}{2}.$$

5. Recall a simple graph G is called *outerplanar* if it can be drawn in the plane so that every vertex is incident with the infinite region.
Let $G = (V, E)$ be a connected outerplanar graph with $|V| \geq 3$.

a) Prove that G contains two vertices of degree at most 2. *(1 point)*

Solution: If G has 3 vertices, then every vertex has the degree at most 2. For an outerplanar G on at least 4 vertices, we prove the following stronger lemma:

Lemma. *G contains at least two non-adjacent vertices of degree at most 2.*

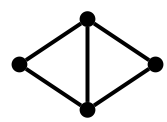
We proceed by induction on n. If the number of vertices is equal to 4, then we know from the lecture that G has at most $2 \cdot 4 - 3 = 5$ edges. In particular, G is not complete and it contains two non-adjacent vertices. Both of these vertices have clearly degree at most 2.

Now suppose $|V| \geq 5$. If G is disconnected, then G has at least two connected components and each component of G contains at least one vertex of degree at most 2 (if the number of the vertices in the component is at most 3, then the statement follows trivially, otherwise we use the induction hypothesis). If G contains a cut-vertex v, then let C_1 be one of the connected components of $G - v$, $V_1 := V(C_1) \cup \{v\}$ and $V_2 := V \setminus V(C_1)$. Let G_1 and G_2 be the subgraph of G induced by V_1 and V_2, respectively. We claim that both G_1 and G_2 contains a vertex $v_1 \neq v$ and $v_2 \neq v$, respectively, of degree at most 2 (clearly, v_i has the same degree in G_i and G for $i \in \{1, 2\}$). Indeed, for both $i = 1$ and $i = 2$, if $|V_i| \leq 3$, then every vertex of G_i has degree at most 2, so just select an arbitrary $v_i \in V_i \setminus \{v\}$. On the other hand, if $|V_i| \geq 4$, then by the induction hypothesis G_i contains two vertices of degree at most 2, so let v_i be one of the vertices that is not v.

It remains to analyze the case that G is 2-connected. Recall from the lecture that 2-connected outerplanar graphs contains a Hamiltonian cycle C. Now, if $E = E(C)$, then G is 2-regular and we can select any two non-adjacent vertices of G. Otherwise, let u and w be two adjacent vertices in G such that $\{u, w\} \notin E(C)$. Clearly, $G - \{u, w\}$ is disconnected, and let C_1 be one of the corresponding connected components. Analogously to the previous case, we set $V_1 := V(C_1) \cup \{u, w\}$ and $V_2 := V \setminus V(C_1)$, and let $G_1 := G[V_1]$ and $G_2 := G[V_2]$. Now we claim for both $i = 1$ and $i = 2$, G_i contains a vertex v_i of degree at most 2 that is neither u nor w. Indeed, if $|V_i| = 3$, then we choose the third vertex in V_i to be v_i. If $|V_i| \geq 4$, then G_i contains at least two non-adjacent vertices x and y of degree at most two. Since u and w are adjacent, it follows that $\{x, y\} \neq \{u, w\}$ and we choose $v_i \in \{x, y\} \setminus \{u, w\}$ arbitrarily.

b) Is it true that G necessarily contains three vertices of degree at most 2? *(1 point)*

Solution: No! See the following graph:



c) Without using the 4-Color Theorem, show that $\chi(G) \leq 3$. *(2 points)*

Solution. The part (a) yields that G is 2-degenerate so indeed, $\chi(G) \leq 3$.

Bonus question. *This question is worth additional* 5 *points on top of the standard* 20 *points.* Show that a graph G is outerplanar if and only if G contains no K_4-minor and no $K_{2,3}$-minor.

Solution. Consider the graph G^+ which is obtained from G by adding a new vertex v and connecting v to all the vertices of G. First of all, if G is outerplanar, we claim that G^+ is planar. Indeed, consider an outerplanar drawing of G in the plane, draw v in the infinite region, and simply connect v to all the vertices of G so that the edges do not cross. We have found a drawing of G^+ so it is planar. However, if G would contain either a minor of K_4 or a minor $K_{2,3}$, we can easily find a minor of K_5 or $K_{3,3}$ in G^+, contradicting Kuratowski's theorem.
Now we essentially flip this argument in order to show the other implication. If G does not contain a minor of K_4 or $K_{2,3}$, then G^+ contains neither a minor of K_5 nor $K_{3,3}$ so by Kuratowski's theorem, G^+ is planar. If a drawing of G^+ is such that v is not on the boundary of the infinite region, then consider any region R with v on its boundary and apply so-called circle inversion to obtain a new drawing of G^+ in the plane so that everything that was drawn outside of R is now inside. In this way, we obtained a drawing D of G^+ with v on the boundary of the infinite region, and it immediately follows that if delete v and all of the edges incident to v from D, we obtain an outerplanar drawing of G. So in particular, G is outerplanar.