MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #00: Additional problems for bonus / practice

Due Tuesday, December 12th, 1:00AM *Write your answers clearly. Justify all your answers.*

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1. For a graph G, let $\kappa(G)$ be the maximum $k \in \mathbb{N}$ such that G is k-connected, and $\kappa'(G)$ the maximum $\ell \in \mathbb{N}$ such that G is ℓ-edge-connected.

For every pair of integers $\ell > k$, construct a simple graph with $\kappa(G) = k$ and $\kappa'(G) = \ell$. *(5 points)*

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2. For a simple graph G, let $\omega(G)$ be the order of a maximum clique in G, and $\chi(G)$ the chromatic number of G.

For every integer $k \geq 2$, prove that exists a constant $C_k > 0$ such that the following is true: if $G = (V, E)$ is a simple graph with $\omega(G) = k$, then $\chi(G) \leq C_k \cdot |V|^{(k-1)/k}$. *(5 points)*

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3. For a simple graph G, we define the *girth* of G to be the length of a shortest cycle contained in G as a subgraph.

Prove that if $G = (V, E)$ is a simple planar graph with girth $g \geq 3$, then $|E| \leq \frac{g}{g-2}|V|$. *(5 points)*

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4. For a simple graph $G = (V, E)$, let $\nu(G)$ be the size of a maximum matching in G, and $\overline{G}$ the complement of G. In other words, $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$.

For a simple triangle-free graph $G = (V, E)$, prove that $\nu(G) + \chi(\overline{G}) = |V|$. *(5 points)*

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5. Decide whether the following is true and either give a proof or provide a counterexampe:
Let $k \geq 2$ be an integer. If G is a $2k$-regular connected graph, then ...

a) ... G must be 2-edge-connected.

b) ... G must be 2-connected. *(5 points in total for (a) and (b))*

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6. A non-negative real matrix is called *doubly stochastic* if both each of its row sums and each of its column sums is 1. A *permutation matrix* is a $(0,1)$-matrix which has exactly one 1 in each row and each column; thus, in particular, every permutation matrix is doubly stochastic.

Prove that every doubly stochastic matrix Q is a square matrix, and can be expressed as a convex combination of permutation matrices. In other words, for some $k \in \mathbb{N}$ and k non-negative reals $\lambda_1, \lambda_2, \ldots, \lambda_k$ with $\sum_i \lambda_i = 1$, we have

$$Q = \sum_{i=0}^{k} \lambda_i \cdot P_i,$$

where $P_1, P_2, \ldots, P_k$ are permutation matrices. *(10 points)*

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7. Let $G = (V, E)$ be a k-connected graph, and let $S \subseteq V$ with $|S| = k$. Prove that G contains a cycle C that goes through all the vertices in S, i.e., $S \subseteq V(C)$. *(10 points)*

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8. For an integer $p \geq 2$, we define a p-uniform hypergraph to be a generalization of (simple) graphs, where the edges connect p different vertices. In other words, a p-uniform hypergraph with the vertex-set V can be described as a pair (V, E) where $E \subseteq \binom{V}{p}$. The complete p-uniform hypergraph on n vertices K_n^p is definied as $\left(V, \binom{V}{p}\right)$ for $V = \{1, 2, \ldots, n\}$.

a) Prove that for every pair of integers $k \geq p$ there exists $n := n(k,p)$ such that the following is true: Every 2-edge-coloring of the edges of K_n^p contains k vertices that contains edges of only one color (i.e., they induce a monochromatic complete hypergraph K_k^p).

b) Using (a), prove that for every $\ell \in \mathbb{N}$ there exists $n := n(\ell)$ such that the following is true: If n points in the plane are in the so-called *general position* (i.e., no three points lie on the same line), then there are ℓ points among them that form the vertices of a convex polygon. *(10 points)*

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9. Let $k \in \mathbb{N}$, let $S_1, S_2, \ldots, S_k$ be a partition of the set $\{1, 2, \ldots N\}$, and let $T_1, T_2, \ldots, T_k$ be arbitrary trees on the vertex-sets $S_1, S_2, \ldots, S_k$, respectively. In other words, $F := \bigcup_{i=1}^{k} T_i$ is a spanning forest of K_N. Prove that K_N has precisely

$$\prod_{i=1}^{k} |S_i| \times N^{k-2}$$

different labelled spanning trees that contain all the edges of F. *(10 points)*

10a) Let G be a bipartite multigraph with a bipartition (A, B) such that $|A| = |B| = n$, where all the vertices have degree 3 except for one vertex $a \in A$ and one vertex $b \in B$, which both have degree 2. Prove that G has at least $2 \cdot (4/3)^{n-1}$ perfect matchings.

10b) Deduce from the part (a) that if G is a 3-regular bipartite multigraph with $2n$ vertices, then G has at least $(4/3)^n$ perfect matchings. *(10 points in total for (a) and (b))*