MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #1: Paths and Cycles

Due Thursday, September 21st, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) For every $k \geq 2$, construct a simple connected graph on $2k$ vertices such that the degree of every vertex is equal to 3. *(0 points)*

b) For every $k \geq 2$, prove that there is no graph on $2k-1$ vertices such that the degree of every vertex is equal to 3. *(0 points)*

c) Let G be a simple graph on $n \geq 2$ vertices. Show that there exist two vertices $u, w \in V(G)$ such that $\deg(u) = \deg(w)$. *(0 points)*

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1. For each of the following statements decide whether it is true (and then prove it) or false (give a counterexample).

a) Let $G = (V, E)$ be a simple graph, and let $\overline{G}$ be the complement of G, i.e., the graph $(V, \overline{E})$, where $\overline{E} := \binom{V}{2} \setminus E$. If G is not connected, then $\overline{G}$ is connected. *(1 point)*

b) Let G be a graph and e, f, g be three edges of G. If there is a cycle in G containing e and f, and a cycle containing f and g, then there is a cycle containing e and g. *(1 point)*

c) Let G be a graph and u, v, w be three vertices of G. If there is a cycle in G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w. *(1 point)*

d) Let G be a connected graph which contains no path of length more than k. Do every two paths in G of length k have at least one vertex in common? *(1 points)*

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2. Let G be a simple graph such that the degree of every vertex is at least k. Prove that G contains a cycle of length at least k. *(3 points)*

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3. Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \ldots, v_n\}$. Recall that the adjacency matrix A_G of G is a symmetric $n \times n$ matrix with the (i, j)-th entry being

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For every integer k and $i, j \in \{1, 2, \ldots, n\}$, prove that the (i, j)-th entry of $(A_G)^k$ is equal to the number of walks of length k in G from v_i to v_j. *(3 points)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

Let G be a simple **connected** graph such that the degree of every vertex is at least k. Prove that G contains a path of length $\min\{2k, |V(G)| - 1\}$. *(0 points)*