MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #6: Matchings in arbitrary graphs, Tutte's theorem

Due Thursday, October 26th, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Show that it is impossible, using 1×2 rectangles (dominoes), to tile an 8×8 square (chesboard) from which the two opposite 1×1 corners ($A1$ and $H8$) have been removed. *(0 points)*

b) Let G be a graph and let M be a maximal matching in G w.r.t. adding edges (i.e., there is no matching M' in G such that $M \subsetneq M'$). Prove that $|M| \geq \lceil \nu(G)/2 \rceil$. *(0 points)*

c) Let G be a graph and let M and N be two matchings in G such that $|M| > |N|$. Prove that there exist matchings M' and N' such that $M \cup N = M' \cup N'$, $|M'| = |M| - 1$ and $|N'| = |N| + 1$. *(0 points)*

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1. Let T be a tree. Prove that T has a perfect matching if and only if for every vertex $v \in V(T)$ the subgraph $T - v$ contains exactly one connected component with odd number of vertices. *(2 points)*

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2. Let G be a 3-regular simple graph with no cut-edge, and let $e \in E(G)$ be an edge of G.

a) Prove that G has a perfect matching M_1 such that $e \in M_1$. *(2 points)*

b) Prove that G has a perfect matching M_2 such that $e \notin M_2$. *(2 points)*

3.

a) Let $k \geq 3$ and $G = (V, E)$ a k-regular connected graph with even number of vertices. Suppose G has the property that for every set of edges $F \subseteq E$ of size $k-2$, the subgraph $(V, E \setminus F)$ is still connected (graphs with this property are called $(k-1)$-*edge-connected*). Prove that G has a perfect matching.

(2 points)

b) For every $k \geq 3$, construct a k-regular graph $Z_k = (V, E)$ with even number of vertices which has the property that for every subset of edges $F \subseteq E$ of size $k-3$, the subgraph $(V, E \setminus F)$ is still connected (i.e., Z_k is $(k-2)$-*edge-connected*), but yet Z_k has no perfect matching. *(2 points)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

a) Prove that a graph $G = (V, E)$ has a 2-factor if and only if for every vertex-subset $X \subseteq V$ and every independent set $S \subseteq (V \setminus X)$ it holds that *(0 points)*

$$|S| \leq |X| + \sum_{C \text{ component of } G-(S\cup X)} \left\lfloor \frac{\#\text{ of edges between } S \text{ and } C}{2} \right\rfloor .$$

Hint: construct an auxiliary graph H s.t. 2-factor in $H \equiv$ 1-factor in G, and use Tutte's theorem for H.

b) Use part (a) to prove that for all $k \geq 1$, a $(2k+1)$-regular graph G with no cut-edge has a 2-factor.

(0 points)