This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Construct a red/blue coloring of $E(K_8)$ such that the coloring contains neither red $K_3$ nor blue $K_4$.

b) Prove that $R(3, 4) = 9$.  

(0 points)

c) Prove that $R(4, 4) \leq 18$.  

(0 points)

[Note that there exists a coloring of $E(K_{17})$ coming from number theory that has no monochromatic $K_4$.]

1. For given integers $k, \ell$ and $m$, recall that $R(k, \ell, m)$ is the smallest integer $N$ such that any red/blue/green coloring of $E(K_N)$ contains at least one of the following subgraphs: a red copy of $K_k$, a blue copy of $K_\ell$, or a green copy of $K_m$. Prove that

$$R(k, \ell, m) \leq \frac{(k + \ell + m - 3)!}{(k - 1)! (\ell - 1)! (m - 1)!}.$$  

(3 points)

2. Let $R_k(3) := R(3, 3, \ldots, 3)$ is the minimum integer $n$ such that any $k$-coloring of $E(K_n)$ contains a monochromatic $K_3$. Prove that $R_k(3) \leq 3k!$ for any integer $k \geq 1$.  

(3 points)

3. A tournament is an oriented graph where every two vertices $u$ and $v$ are joined by either an oriented edge from $u$ to $v$ ($u \to v$), or from $v$ to $u$. More formally, it is a triple $(V, E, o)$ where $V$ is a set of vertices, $E = \binom{V}{2}$ a set of edges, and $o : E \to V$ such that $o(\{u, v\}) \in \{u, v\}$ a function determining the orientation by letting $o(e)$ to be the source of the edge $e \in E$.

An oriented cycle in a tournament $T = (V, E, o)$ is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ where $v_i \in V$, $e_i \in E$, $v_1 = v_{k+1}$, $e_i = \{v_i, v_{i+1}\}$ and $o(e_i) = v_i$ for all $i \in \{1, 2, \ldots, k\}$. A vertex-subset $U$ of $V$ is called acyclic set in $T$ if the subtournament of $T$ induced by $U$ contains no oriented cycle.

Let $T$ be an $n$-vertex tournament. Prove that $T$ contains an acyclic set of size $\lceil \log_2(n) \rceil + 1$.  

(4 points)

(*) This is a challenge of the week question, do not submit your solution.

Let $G_n$ be the set of all the $n$-vertex simple graphs that contain no triangle.

a) For all but finite number of values of $n \in \mathbb{N}$, prove there exists $G \in G_n$ with $\alpha(G) \leq n^{0.99}$. Can you do even better than 0.99 in the exponent? Say $\alpha(G) = O\left(n^{2/3}\right)$? Or even $\alpha(G) = O\left(n^{0.51}\right)$? (0 points)

[The current record in this direction is: for every $\varepsilon > 0$ there is $n_0$ such that for all $n \geq n_0$, there exists $G \in G_n$ with $\alpha(G) < (1 + \varepsilon) \cdot \sqrt{2n \log(n)}$.]

b) Prove there exists a constant $c > 0$ such that every $G \in G_n$ has $\alpha(G) > c \cdot \sqrt{n \log(n)}$.  

(0 points)

[The current record in this direction is: $\alpha(G) > \sqrt{\frac{1}{2} \cdot n \log(n)}$ for all $G \in G_n$.]