MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #8: Connectivity & Menger's theorem, Network flows

Due Tuesday, November 14th, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Let $G = (V, E)$ be a 2-connected graph and $u, v \in V$. Prove G has a cycle $C_{u,v}$ going through u and v. *(0 points)*

b) Let $G = (V, E)$ be a **2-connected (2-edge-connected was of course not enough here)** graph and $e, f \in E$. Prove G has a cycle $C_{e,f}$ going through e and f. *(0 points)*

c) Let G be a 3-regular graph. For every $k \in \{0, 1, 2, 3\}$, prove that G is k-connected if and only if G is k-edge-connected. *(0 points)*

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1. Let $G = (V, E)$ be a 2-connected graph and $v \in V$ a vertex of G. Prove that there exists a vertex $u \in V$ such that $\{u, v\} \in E$ and the graph $G - u - v$ is connected. *(3 points)*

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2. Let $H = (V, E)$ be a graph and let $U \subseteq V$. We define $H \oplus_U \{v\}$ to be the graph obtained from H by adding a new vertex v, which is then joined to every vertex in U. In other words,

$$H \oplus_U \{v\} = (V \cup \{v\}, E \cup \{\{u, v\} : u \in U\}) .$$

a) Prove that if $G = (V, E)$ is a k-connected graph and $U \subseteq V$ has size k, then the graph $G \oplus_U \{v\}$ is k-connected as well. *(2 points)*

b) Let $G = (V, E)$ be a k-connected graph and $U, W \subseteq V$ two vertex-subsets, each of size k. Prove that there exist k pairwise vertex-disjoint paths $P_1, \ldots P_k$ such that for every $i \in \{1, \ldots, k\}$, the path P_i have one endpoint in U and the other endpoint in W. *(1 point)*

c) Let $G = (V, E)$ be a 2-connected graph. Show that for any triple of distinct vertices $u, v, w \in V$ there is a path in G from u to v passing through w, i.e., w is an inner-vertex of the path. *(1 point)*

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3. Let $G = (V, E)$ be a directed graph (digraph) and for each edge $e \in E$, let $\phi(e) \geq 0$ be a non-negative integer. Show that if for every vertex v

$$\sum_{e \in \partial^-(v)} \phi(e) = \sum_{e \in \partial^+(v)} \phi(e) ,$$

then there is a collection of directed cycles $C_1, ..., C_k$ (possibly with repetition) so that for every edge e of G, it holds that $|\{i : 1 \leq i \leq k,\ e \in E(C_i)\}| = \phi(e)$. *(3 points)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

Prove that for every $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that the following is true: If $G = (V, E)$ is an ℓ-connected graph and $U, W \subseteq V$ two vertex-subsets of size k with $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$, then there exist k vertex-disjoint paths in G, where the i-th path has endpoints u_i and w_i. *(0 points)*