Instructions: The exam is 3 hours long and contains 6 questions. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. Justify all your answers.

Q1 Let $G$ be the graph pictured on Figure 1.
   a) Is $G$ planar?
   b) Find $\nu(G)$ and $\tau(G)$.
   c) Find $\chi(G)$ and $\chi'(G)$.

Q2 Let $G$ be the graph with weights $w : E(G) \to \mathbb{Z}_+$ pictured on Figure 2.
   a) Find the min-cost spanning tree in $G$.
   b) Find a shortest path spanning tree for the vertex $A$.

Q3 Let $k \geq 3$ be an integer. Let $G$ be a bipartite graph such that
   $$3 \leq \deg(v) \leq k \quad \text{for every } v \in V(G).$$
   Show that $G$ contains a matching of size at least $\frac{3|V(G)|}{2k}$.

Q4 Let $G$ be a loopless graph, such that $G$ does not contain $K_{2,3}$ as a minor. Show that either $\chi(G) \leq 3$, or $G$ contains $K_4$ as a subgraph.

Q5 Let $G$ be a non-planar graph such that every subgraph of $G$, except for $G$ itself, is planar. Show that $|E(G)| - |V(G)| = 3$, or $|E(G)| - |V(G)| = 5$.

Q6 Let $G$ be a simple graph with $|V(G)| \geq 2$. Suppose that $G$ does not contain $P_4$ (the path on 4 vertices) as an induced subgraph.
   a) Prove that either $G$ is not connected or the complement of $G$ is not connected. (Hint: Use induction on $|V(G)|$. Show that, if $G \setminus v$ has at least two components and $v$ is adjacent to a vertex in every component of $G \setminus v$, then $v$ is adjacent to every vertex of $G \setminus v$.)
   b) Deduce from a) that $G$ is perfect. You are allowed to use the weak perfect graph theorem, but not the strong perfect graph theorem.
Figure 1: The graph in the question Q1.

Figure 2: The graph in the question Q2.