Instructions: The exam is 3 hours long and contains 6 questions. The total number of points is 100. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. Justify all your answers!

Q1 Let $G$ be the graph depicted in Figure 1.

![Figure 1: The graph in the question Q1.](image1)

a) Is $G$ planar? (4 points)

Solution: Yes. See the following drawing of $G$.

![Figure 2: The graph in the question Q2.](image2)
b) Find $\nu(G)$ and $\tau(G)$.

Solution: $G$ has 9 vertices, so $\nu(G) \leq \left\lfloor \frac{9}{2} \right\rfloor = 4$. On the other hand, the red edges on the picture below shows a matching of size 4 in $G$. We already know that $\tau(G) \geq \nu(G) = 4$. The blue vertices on the picture below shows a vertex cover of size 5; in the rest we show that $\tau(G) > 4$. Suppose there is a vertex cover $X$ in $G$ of size 4. Then $X$ must contain the middle vertex $v$ of $G$, otherwise $X$ needs to contain all the other eight vertices. However, that implies that the set $X - v$ is a vertex cover of size 3 in the graph $G - v \supset C_8$. But that is a contradiction (for example recall that $\alpha(C_8) = 4$ and $\alpha(H) + \tau(H) = |V(H)|$ for any graph $H$).
c) Find $\chi(G)$.  

**Solution:** Since $G$ contains $K_4$ as a subgraph $\implies \chi(G) \geq 4$. See on the left picture below a 4-coloring of $G$ which shows $\chi(G) \leq 4$.

d) Find $\chi'(G)$.  

**Solution:** The middle vertex has degree 8 $\implies \chi'(G) \geq 8$. See on the right picture below an 8-edge-coloring of $G$ which shows $\chi'(G) \leq 8$.

Figure 2: A 4-coloring of $G$ from Q1 with red, blue, green and yellow. An 8-edge-coloring of $G$ from Q1 with red, blue, green, yellow, orange, gray, purple and brown.
Q2 Let $\vec{G} = (V, E)$ be the oriented graph with the two specific vertices $s$ and $t$ and with the capacities $c : E \to \mathbb{Z}_+$ depicted in Figure 3.

Figure 3: The oriented graph in the questions Q2 and Q3.

a) Find a maximum flow from the vertex $s$ to the vertex $t$. (8 points)

Solution: See the following flow from $s$ to $t$ of value 6.
b) Find a minimum $s,t$-cut. \hspace{1cm} (8 points)

**Solution:** See the following $s,t$-cut of capacity 6.

![Graph with marked s,t-cut]

The construction of a flow from $s$ to $t$ of value 6 in (a) and the construction of an $s,t$-cut with capacity 6 in (b) yields that the flow is a maximum flow and the $s,t$-cut has minimum capacity. Note that the constructed max-flow / min-cut could be obtained by iteratively applying Ford-Fulkerson Theorem (see Theorem 10.5 in the lecture notes).
Let \( G = (V, E) \) be the simple graph with weights \( w : E \to \mathbb{Z}_+ \) obtained from the oriented graph depicted in Figure 3 by replacing each oriented edge by a non-oriented one that has the same weight.

a) Find a shortest path spanning tree in \( G \) for the vertex \( S \). \( (6 \text{ points}) \)

Solution: See the following shortest path spanning tree for \( S \): 

Note that this tree can be obtained using Dijkstra’s algorithm (see Chapter 5 in the lecture notes).
b) Find a minimum-cost spanning tree in $G$.  

   \textbf{Solution:} See a minimum spanning tree of the total weight 6:

   \begin{center}
   \begin{tabular}{c|c|c|c|c}
   S & T & A & B & C \\
   1 & 2 & 3 & 1 & 3 \\
   D & E & 2 & 1 & 1 \\
   \end{tabular}
   \end{center}

   Note that this tree can be obtained using Kruskal’s algorithm (see Chapter 4 in the lecture notes).

c) Does $G$ have a unique minimum-cost spanning tree?  

   \textbf{Solution:} No! See another spanning tree of the total weight 6:

   \begin{center}
   \begin{tabular}{c|c|c|c|c}
   S & T & A & B & C \\
   1 & 2 & 3 & 1 & 3 \\
   D & E & 2 & 1 & 1 \\
   \end{tabular}
   \end{center}
Q4 Let $k \geq 2$ be an integer, and let $G$ be a connected $k$-regular bipartite graph. Prove that $G$ is 2-connected. (16 points)

**Solution:** Let $(A, B)$ be a bipartition of $G$. Suppose for contradiction that $G$ contains a cut-vertex $v$. Without loss of generality, $v \in A$. Let $(A_1, B_1)$ be a bipartition of one of the components of $G - v$ with $A_1 \subseteq A$ and $B_1 \subseteq B$, and let $B_0 := B \setminus B_1$. Let $\ell_0$ and $\ell_1$ be the number of neighbors of $v$ in $B_0$ and $B_1$, respectively. Since $G$ is connected, then $\ell_1 \geq 1$ and hence

$$1 \leq \ell_0 = k - \ell_1 \leq k - 1.$$ 

Therefore, the number of edges $z$ from $B_1$ to $A_1$ in $G - v$ is $|B_1|k - \ell_0$. In particular, $z \not\equiv 0 \mod k$. On the other hand, every vertex in $A_1$ has degree $k$, so $z \equiv 0 \mod k$; a contradiction.
Q5 Let $G$ be a simple planar graph. Prove that if $G$ contains no cycle of length five or less, then $\chi(G) \leq 3$.\(^{16\text{ points}}\)

**Solution:** We prove the statement by induction on $|V(G)|$. If $|V(G)| \leq 3$, then the statement is indeed trivially true. In the rest, we assume $|V(G)| \geq 4$.

If $G$ is disconnected, we can use the induction hypothesis on each of its components to find a 3-coloring of each component. The union of these 3-colorings yields a 3-coloring of the whole graph $G$.

Now suppose $G$ is connected and has a cut-vertex $x$. Let $C$ be one of the connected components of $G - x$ and let $V_1 := V(C) \cup \{x\}$ and $V_2 := V(G) \setminus V(C)$. It follows from the induction hypothesis that the two subgraphs $G_1$ and $G_2$ induced by $V_1$ and $V_2$, respectively, are both 3-colorable. Moreover, we can find a 3-coloring $c_1$ of $G_1$ and a 3-coloring $c_2$ of $G_2$ such that the vertex $x$ is colored both in $c_1$ and $c_2$ with the color 1. But then the union of these two 3-colorings again yields a 3-coloring of the whole graph $G$.

For the rest of the proof, suppose that $G$ is 2-connected and consider any drawing of $G$. Since $G$ is 2-connected, the boundary of any region $R$ in the drawing consists of a cycle in $G$. But $G$ has no cycles of length at most 5, so $\ell(R) \geq 6$. Therefore,

$$2|E(G)| = \sum_{R \text{ region}} \ell(R) \geq 6 \cdot \text{Reg}(G).$$

This yields that $\text{Reg}(G) \leq |E(G)|/3$ and from the Euler’s formula we conclude

$$2 = |V(G)| + \text{Reg}(G) - |E(G)| \leq |V(G)| - 2|E(G)|/3.$$

A basic algebra manipulation with the last inequality yields that

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| \leq 3|V(G)| - 6,$$

which in-turn implies that $G$ must contain a vertex $x$ of degree 2. By the induction hypothesis applied on $G - x$, we can find a 3-coloring of $G - x$, which is then easily extended to $G$ by coloring $x$ with any color that is not present on its two neighbors.
Q6 Let $K^-_4$ be the 4-vertex graph obtained from $K_4$ by removing one edge. How many non-isomorphic simple 2-connected graphs $G = (V, E)$ are there with $|V| = 1000$ such that $G$ has no $K^-_4$-minor? (18 points)

Solution: The answer is 1 and the only graph $G$ is $C_{1000}$. If $G$ is 2-connected, then it must contain a cycle. Let $C$ be the longest cycle in $G$. If $G$ is not isomorphic to $C_{1000}$, then either $C$ is a Hamilton cycle and $G$ contains at least one chord of $C$, or, the length of $C$ is at most 999. If $C$ is a Hamilton cycle and $G$ contains at least one chord of $C$, then clearly $C$ and this chord is a minor of $K^-_4$.

On the other hand, if $C$ does not cover all the vertices, then let $x \in V(G) \setminus V(C)$. By the 2-connectivity assumption on $G$, it contains two paths $P_1$ and $P_2$ so that $V(P_1) \cap V(P_2) = \{x\}$, $V(C) \cap V(P_1) = \{w_1\}$, $V(C) \cap V(P_2) = \{w_2\}$, the endpoints of $P_1$ are $x$ and $w_1$, and the endpoints of $P_2$ are $x$ and $w_2$. But then $C \cup P_1 \cup P_2$ is a minor of $K^-_4$. 