Instructions: The exam is 3 hours long and contains 6 questions. The total number of points is 100. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. Justify all your answers!

Q1 Let $G$ be the graph depicted in Figure 1.

   a) Is $G$ planar? (4 points)
   b) Find $\nu(G)$ and $\tau(G)$. (4 points)
   c) Find $\chi(G)$. (4 points)
   d) Find $\chi'(G)$. (4 points)

Q2 Let $\overrightarrow{G} = (V, E)$ be the oriented graph with the two specific vertices $s$ and $t$ and with the capacities $c : E \to \mathbb{Z}_+$ depicted in Figure 2.

   a) Find a maximum flow from the vertex $S$ to the vertex $T$. (8 points)
   b) Find a minimum $S,T$-cut. (8 points)

Q3 Let $G = (V, E)$ be the simple graph with weights $w : E \to \mathbb{Z}_+$ obtained from the oriented graph depicted in Figure 2 by replacing each oriented edge by a non-oriented edge with the same weight.

   a) Find a shortest path spanning tree in $G$ for the vertex $S$. (6 points)
   b) Find a minimum-cost spanning tree in $G$. (6 points)
   c) Does $G$ have a unique minimum-cost spanning tree? (6 points)

Q4 Let $k \geq 2$ be an integer, and let $G$ be a connected $k$-regular bipartite graph. Prove that $G$ is 2-connected. (16 points)

Q5 Let $G$ be a simple planar graph. Prove that if $G$ contains no cycle of length five or less, then $\chi(G) \leq 3$. (16 points)

Q6 Let $K_4^-$ be the 4-vertex graph obtained from $K_4$ by removing one edge. How many non-isomorphic simple 2-connected graphs $G = (V, E)$ are there with $|V| = 1000$ such that $G$ has no $K_4^-$-minor? (18 points)
Figure 1: The graph in the question Q1.

Figure 2: The oriented graph in the questions Q2 and Q3.