Instructions: The exam is 3 hours long and contains 6 questions. The total number of points is 100. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. **Justify all your answers!**

Q1 Let $G$ be the graph depicted in Figure 1.

a) Is $G$ planar? (4 points)
b) Does $G$ contain a Hamilton cycle? (4 points)
c) Find $\chi(G)$. (4 points)
d) Find $\chi'(G)$. (4 points)

Q2 Let $\vec{G} = (V,E)$ be the oriented graph with the two specific vertices $s$ and $t$ and with the capacities $c : E \to \mathbb{Z}_+$ depicted in Figure 2.

a) Find a maximum flow from the vertex $s$ to the vertex $t$. (8 points)
b) Find a minimum $s,t$-cut. (8 points)

Q3 Let $G = (V,E)$ be the simple graph with weights $w : E \to \mathbb{Z}_+$ obtained from the oriented graph depicted in Figure 2 by replacing each oriented edge by a non-oriented edge with the same weight.

a) Find a shortest path spanning tree in $G$ for the vertex $s$. (6 points)
b) Find a minimum-cost spanning tree in $G$. (6 points)
c) Does $G$ have a unique minimum-cost spanning tree? (6 points)

Q4 Let $G = (V,E)$ be a graph, and let $u,v,w \in V$ be distinct.

a) Suppose $G$ contains two internally vertex-disjoint paths from $u$ to $v$ and two internally vertex-disjoint paths from $v$ to $w$. Prove or disprove that $G$ has to contain two internally vertex-disjoint paths from $u$ to $w$? (8 points)
b) Suppose $G$ contains two edge-disjoint paths from $u$ to $v$, and two edge-disjoint paths from $v$ to $w$. Prove or disprove that $G$ has to contain two edge-disjoint paths from $u$ to $w$? (8 points)

Q5 Let $G = (V,E)$ be an $n$-vertex bipartite graph such that $3 \leq \deg_G(v) \leq k$ for every $v \in V$. Prove that $G$ contains a matching of size at least $\frac{3n}{k+3}$. (17 points)

Q6 Let $G$ be an $n$-vertex graph with $\alpha(G) = 2$. Prove that $G$ contains $K_{\left[\frac{n}{2}\right]}$ as a minor. (17 points)
Figure 1: The graph in the question Q1.

Figure 2: The oriented graph in the questions Q2 and Q3.