Graph Theory & Combinatorics
McGill University, Fall 2012

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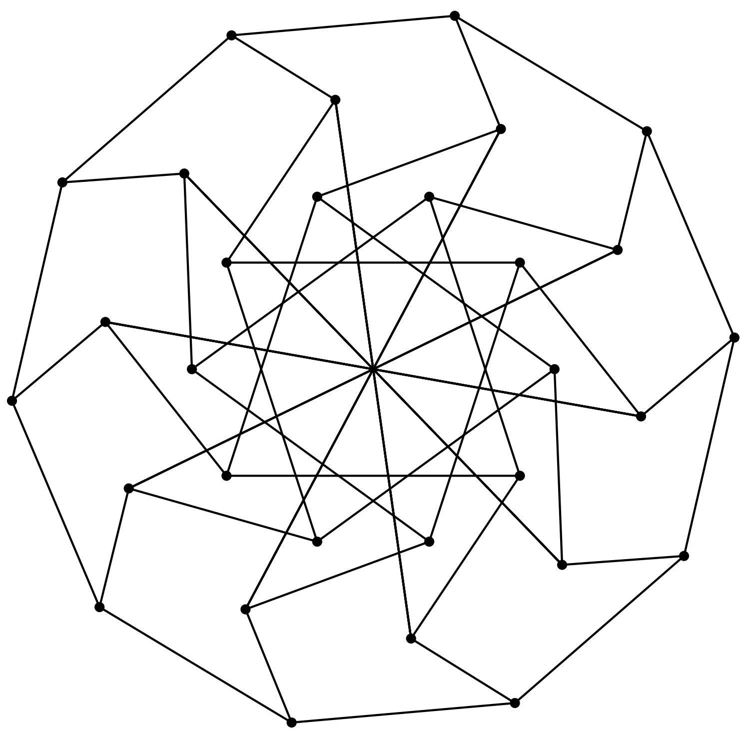

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1 Introduction

As a disclaimer, these notes may include mistakes, inaccuracies and incomplete reasoning. On that note, we begin.

Graph theory is the study of dots and lines: sets and pairwise relations between their elements.

Definition. A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a set of *vertices*, $E(G)$ is a set of *edges*, and a edge is said to be *incident* to one or two vertices, called its *ends*. If e is incident to vertices u and v, we write $e = uv = vu$.

When $V(G)$ and $E(G)$ are finite, G is a *finite graph*. In this course, we only study and consider finite graphs.

Definition. A *loop* is an edge with only one end.

Definition. Edges are said to be *parallel* if they are incident to the same two vertices.

Definition. A graph with no loops and no parallel edges is called *simple*. On the other hand, in order to emphasize that a graph is not simple we call it a *multigraph*.

Definition. A *loopless* (multi)graph is a (multi)graph without loops.

Definition. Two distinct vertices are *adjacent* if they are ends of some edge. In this case, one is a *neighbour* of the other. Two distinct edges are *incident* if have a common end.

Definition. A *null graph* is a graph with no vertices and no edges.

Definition. A *complete graph* on n vertices is denoted K_n, and is a simple graph in which every two vertices are adjacent.

Definition. A *path* on n vertices, denoted P_n, is a graph such that:

$$V(P_n) = \{v_1, v_2, \dots, v_n\}$$
$$E(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$$

where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$.

Definition. A *cycle* on n vertices, denoted C_n, is a graph such that:

$$V(C_n) = \{v_1, v_2, \dots, v_n\}$$
$$E(C_n) = \{e_1, e_2, \dots, e_n\}$$

where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$, and $e_n = v_n v_1$.

Definition. The *length* of a path or a cycle is its number of edges.

Definition. A *walk* in a graph G is a non-empty alternating sequence $v_0e_1v_1e_1 \dots e_kv_k$ of vertices and edges in G, such that $e_i = v_{i-1}v_i$. If $v_0 = v_k$, then the walk is said to be *closed*.

Definition. A walk $v_0e_1v_1e_1 \dots e_kv_k$ in a graph G is called a trail in G if $e_i \neq e_j$ for every $1 \leq i < j \leq k$. A trail in G is called *closed* if $v_0 = v_k$, i.e., if it is a closed walk.

Definition. The *degree* of a vertex v in a graph G is the number of edges incident to it (with loops counted twice). This is denoted $\deg_G(v)$, or sometimes simply $\deg(v)$ when G is understood.

Theorem 1.1. *For any graph G, we have:*

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Proof. Summing the degrees of every vertex, each edge is counted exactly twice. □

Definition. A graph H is a *subgraph* of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. The edges of H have the same ends in G. We will write that $H \leq G$, or $H \subset G$.

Remark. If $H_1, H_2 \leq G$, then $H_1 \cup H_2 \leq G$, with:

$$V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$$
$$E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$$

$H_1 \cap H_2$ is defined analogously.

Definition. A *path* of G is a subgraph of G which is a path.

2 Connectivity

Lemma 2.1. *Let G be a graph which is not connected. Then there exists a partition (X, Y) of G, $X, Y \neq \varnothing$, such that no edge of G has one end in X and another in Y.*

Definition. A graph G is connected if and only if for all $u, v \in V(G)$, there exists a walk from u to v.

Lemma 2.2. *Let G be a connected graph, let $X \subsetneq V(G)$ with $X \neq \varnothing$. Then there exists an edge of G with one edge in X and another in $V(G) - X$.*

Lemma 2.3. *If there is a walk with ends u, v in G, then there is a path in G with the same ends.*

Proof. Choose the walk in G with ends u and v of minimal length. We will show that it corresponds to a path, i.e. that it has no repeated vertices.

Suppose not, i.e. the walk is given by $v_0 e_1 v_1 \ldots v_n$, and $v_i = w_j$ for some $i < j$. Then $v_0 e_1 v_1 \ldots v_i \ldots v_j \ldots v_n$ is a shorter walk with the same ends, which proves the claim by contradiction. □

Lemma 2.4. *If H_1, H_2 are connected subgraphs of G so that $H_1 \cap H_2 \neq \varnothing$, then $H_1 \cup H_2$ is connected.*

Proof. Let $w \in V(H_1) \cap V(H_2)$. Now, for all $u, v \in V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$. there exists a walk from u to w which lies entirely WLOG in H_1, and a walk from w to v lying WLOG entirely in H_2. Concatenating these walks, the result is immediate. □

Definition. A *connected component* of a graph G is a maximal connected subgraph.

Lemma 2.5. *In a graph G, every vertex is in a unique connected component.*

Proof. Suppose $v \in C_1, C_2$, where C_1 and C_2 are distinct connected components. Then, by Lemma 2.4, since $v \in C_1 \cap C_2 \neq \varnothing$, then $C_1 \cup C_2$ is connected, contradicting the maximality of C_1 and C_2. □

We let $\text{comp}(G)$ denote the number of connected components of G. From the previous lemma, this notion is well-defined.

Definition. An edge is a *cut edge* of G if it belongs to no cycle of G.

Lemma 2.6. *Let G be a graph, and let F be another graph obtained by adding an edge e to G. Then, either:*

a) e is a cut edge of F, in which case u and v are in different components of G, and:

$$comp(F) = comp(G) - 1$$

b) e is not a cut edge of F, in which case u and v are in the same component of G, and:

$$comp(F) = comp(G)$$

Let G be a graph, $e \in E(G)$. We say that a subgraph H of G is *obtained by deleting e* if $V(H) = V(G)$ and $E(H) = E(G) - \{e\}$. We denote this subgraph by $G \setminus e$.

If $v \in V(G)$, we say that H is *obtained by deleting v* if $V(H) = V(G) - \{v\}$, and $E(H) = E(G) - \{(a,b) : a = v \text{ or } b = v\}$. It is denoted by $G \setminus v$.

Lemma 2.7. *Let $e \in E(G)$. Then either:*

a) e is a cut edge of G, u and v lie in different components of $G \setminus e$, and:

$$comp(G \setminus e) = comp(G) + 1$$

b) e is not a cut edge of G, u and v lie in the same component of $G \setminus e$, and:

$$comp(G \setminus e) = comp(G)$$

3 Trees and Forests

Definition. A *forest* is a graph with no cycles.

Definition. A *tree* is a non-null connected forest.

Theorem 3.1. *If G is a non-null forest, then:*

$$comp(G) = |V(G)| - |E(G)|$$

Proof. We induct on $|E(G)|$. When $|E(G)| = 0$, G is a disconnected set of vertices, and $\text{comp}(G) = |V(G)|$. Otherwise, let $e \in E(G)$. Since G is a forest, then e is a cut-edge, so by the induction hypothesis:

$$\text{comp}(G) = \text{comp}(G \setminus e) - 1 = |V(G)| - (|E(G)| - 1) - 1 = |V(G)| - |E(G)|$$

□

Corollary. *If T is a tree, then:*

$$|V(T)| = |E(T)| + 1$$

Definition. A vertex v in a tree with $\deg(v) = 1$ is called a *leaf*.

Lemma 3.2. *Let T be a tree, $|V(T)| \geq 2$. Let X be the set of leaves of T, and let Y be the set of vertices of degree ≥ 3. Then, $|X| \geq |Y| + 2$, and in particular, $|X| \geq 2$.*

Proof. Let n_i denote the number of vertices of degree i in T. Then:

$$|V(T)| = \sum_i n_i$$

By Theorem 1.1:

$$2|E(T)| = \sum_i i n_i$$

By Theorem 3.1, $2|V(T)| = 2|E(T)| + 2$. Moreover, $n_0 = 0$, and $n_1 = |X|$, so:

$$\sum_i 2n_i = 2 + \sum_i i n_i$$
$$n_1 = |X| = 2 + n_3 + 2n_4 + \ldots + (d-2)n_d \geq 2 + |Y|$$

□

Lemma 3.3. *Let T be a tree with $|V(T)| \geq 2$. Then, T has at least 2 leaves, and if T has exactly 2 leaves, then T is a path.*

Proof. It is enough to show the last statement.

Let a, b be the two leaves of T. By Lemma 2.3, there is a path P from a to b in T. By Lemma 3.2, no vertex in $V(P)$ is incident to an edge of $E(T) - E(P)$. As P is connected, either $V(P) = V(T)$, or there is a edge with exactly one end in $V(P)$. As we know that this second option is impossible, then $V(P) = V(T)$, and then $E(T) = E(P)$. □

Lemma 3.4. *Let T be a tree, with $|V(T)| \geq 2$. Let $v \in V(G)$ be a leaf. Then, $T \setminus v$ is a tree.*

Proof. $T \setminus v$ is non-null and without cycles. It is connected, as no path between two vertices in $V(T \setminus v)$ can include v, so all paths are preserved. Otherwise, since the edge containing v is a cut edge, then $T \setminus v$ has 1 component and is thus connected. □

Lemma 3.5. *Let G be a graph, v a vertex of degree 1 in G. If $G \setminus v$ is a tree, then so is G.*

Proof. G is non-null and has no cycle, so G is a forest. Then:

$$1 = \text{comp}(G \setminus v) = |V(G \setminus v)| - |E(G \setminus v)| = |V(G)| - |E(G)| = \text{comp}(G)$$

So G has 1 component and G is a tree. □

Lemma 3.6. *Let T be a tree and $a, b \in V(T)$. Then, there exists exactly one path with ends a, b in T.*

Proof. By Lemma 3.3, there exists at least one path. Suppose P_1 and P_2 are two distinct paths. Then, $P_1 \cup P_2$ is a tree. But every vertex except a, b has degree 2 in P_1 or P_2, so $P_1 \cup P_2$ has at most 2 leaves, so by Lemma 3.3 it is a path with ends a and b. So $P_1 = P_2$, which is a contradiction. □

4 Spanning Trees

Definition. Let G be a graph, $T \leq G$ a tree, with $V(G) = V(T)$. Then, T is called a *spanning tree* of G.

Lemma 4.1. *Let G be a connected non-null graph. Let $H \leq G$ be minimal so that $V(H) = V(G)$ and is connected. Then H is a spanning tree of G.*

Proof. We only need to show that H is acyclic. If C is a cycle in H, $e \in E(C)$, then $H \setminus e$ is connected by Lemma 2.7, contradicting the choice of H. □

Lemma 4.2. *Let G be a non-null connected graph. Let $H \leq G$ such that $V(H) = V(G)$, H has no cycles, and H is maximal with these properties. Then, H is a spanning tree of G.*

Proof. Suppose H is not connected. Then, by Lemma 2.1, there exists $X \subset V(G)$, $X \neq \emptyset$, $X \neq V(G)$, such that no edge of H has one end in X and another in $V(G) - X$. On the other hand, G has an edge e with one end $u \in X$ and another $v \in V(G) - X$. Consider H' obtained by adding e to H.

By Lemma 2.7, as u and v are in different components of H, e is a cut edge of H', so H' has no cycles. □

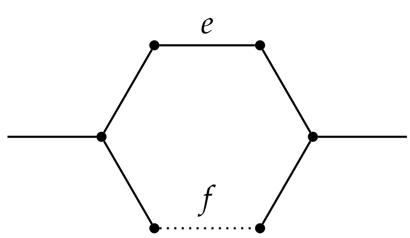
Definition. Let G be a graph, T a spanning tree of G. Let $f \in E(G) - E(T)$. Let C be a cycle of G such that $C \setminus f$ is a path in T. Then C is called a fundamental cycle of f w.r.t. T.

Lemma 4.3. *Let T be a spanning tree of G. Let $f \in E(G) - E(T)$. Then there exists a unique fundamental cycle of f w.r.t. T.*

Proof. The proof follows from Lemma 3.6. □

Lemma 4.4. *Let T be a spanning tree of G, $f \in E(G) - E(T)$. Let C be the fundamental cycle of f, and let $e \in E(C) - \{f\}$. Then $(T + f) \setminus e$ is still a spanning tree of G.*

Proof.



$(T + f) \setminus e$ is connected because e belongs to a cycle C in $T + f$. $(T + f) \setminus e$ has no cycles because C was unique. □

Definition. Let G be a graph. Let $w : E(G) \to \mathbb{R}^+$. A spanning tree T is called a *min-cost* spanning tree (MST) of G if:

$$\sum_{e \in E(G)} w(e)$$

is minimal.

Corollary 4.5. *Let T be a MST for a graph G, with weight function w. Let f, c, e be as in the statement of Lemma 4.4. Then, $w(f) \geq w(e)$.*

Proof. Let $T' = (T + f) \setminus e$. Then:

$$\sum_{e' \in E(T)} w(e') \leq \sum_{e' \in E(T')} w(e') \Rightarrow w(e) \leq w(f)$$

□

Theorem 4.6. *Let G be a graph, w be a weight function, T be an MST for G and w. Assume, for convenience, that $w(e)$ are all distinct for $e \in E(G)$. Let $|V(G)| = n$, and let $e_1, e_2, \ldots, e_{n-1}$ be all the edges of T, with $w(e_1) < w(e_2) < \ldots < w(e_{n-1})$. Then, e_i is the edge of G with $w(e_i)$ minimum subject to the conditions $e_i \notin \{e_1, e_2, \ldots e_{i-1}\}$ and $\{e_1, e_2, \ldots, e_{i-1}, e_i\}$ contains no cycle.*

Proof. Suppose not, for some e_i, $1 \leq i \leq n-1$. Thus, there is an $f \in E(G) - \{e_1, e_2, \ldots e_{i-1}\}$, so that $w(f) < w(e_i)$ and $\{e_1, \ldots, e_{i-1}\} \cup \{f\}$ do not form a cycle. Also, $f \notin E(T)$, because $f \notin \{e_1, e_2, \ldots, e_{i-1}\}$ by assumption, and $f \notin \{e_i, e_{i+1}, \ldots, e_n\}$ because $w(f) < w(e_i) < w(e_j)$ for all $j > i$.

Let C be the fundamental cycle of f w.r.t. T. $E(C) \not\subset \{e_1, e_2, \ldots, e_{i-1}, f\}$ by assumption. On the other hand, by the previous corollary, $w(e) \leq w(f)$ for every $e \in E(C)$, so $E(C) \cap \{e_i, e_{i+1}, \ldots, e_{n-1}\} = \varnothing$, which is a contradiction. □

Kruskal's Algorithm

input: G, $\{w(e) : e \in E(G)\}$

algorithm:

For $1 \leq i \leq n$, if $\{e_1, e_2, \ldots, e_{i-1}\}$ are already defined, let e_i be the edge with $w(e_i)$ minimum, so that $e_i \notin \{e_1, e_2, \ldots, e_{i-1}\}$ and $\{e_1, e_2, \ldots e_{i-1}\} \cup \{e_i\}$ contains no cycle.

output:

By Theorem 4.6, the algorithm outputs $\{e_1, e_2, \ldots, e_{n-1}\}$, the edges of MST($G$).

5 Shortest Paths

Let G be a graph, $w : E(G) \to \mathbb{R}^+$. We are interested in finding, for given vertices u and v, the shortest path P in G from u to v, i.e. a path with $w(P) := \sum_{e \in E(P)} w(e)$ minimized.

Let $\text{dist}_G(u, v)$ denote the length (weight) of the shortest path from u to v.

Definition. Let $s \in V(G)$, $T \leq G$ be a tree containing s (not necessarily spanning). T is called a *shortest path tree* for s if for $v \in V(T)$, $\text{dist}_T(s, v) = \text{dist}_G(s, v)$.

Theorem 5.1. *Let G, w be as in the definition above. Importantly, $w(e)$ is non-negative for all $e \in E(G)$. Let T be a shortest path tree for s. Among all edges e with ends $u \in V(T)$ and $v \notin V(T)$, choose one with $w(e) + dist_G(s, u)$ is minimum. Then, T' obtained from T by adding e and v is a shortest path tree.*

Proof. It suffices to check that $\text{dist}_{T'}(s, v) = \text{dist}_G(s, v)$.

Let P be a shortest path in G from s to v. Let $e' = xy$ be the first edge of P (starting at s), with $y \notin V(T)$. Then:

$$\begin{aligned} w(P) &\geq \text{dist}_G(s, x) + w(e') = \text{dist}_T(s, x) + w(e') \\ &\geq \text{dist}_T(s, u) + w(e) = \text{dist}_{T'}(s, v) \end{aligned}$$

□

Dijkstra's Algorithm

input: Connected graph G, $s, t \in V(G)$, $w(e) \geq 0$ for $e \in E(G)$.

algorithm:

Recursively construct trees $T_1, T_2, \ldots \leq G$, $V(T_1) = s$, $E(T_1) = \varnothing$. If $T_1, T_2, \ldots, T_i$ were constructed, then let e, u, v be chosen with $v \in V(G) - V(T_i)$, $u \in V(T_i)$, with $\text{dist}_{T_i}(s, u) + w(e)$ minimum, let $T_{i+1} = T_i + e$. Stop when $t \in V(T_i)$. The shortest path is then the unique $s - t$ path in T_i.

output: Shortest path in G from s to t.

6 Euler's Theorem and Hamiltonian Cycles

Lemma 6.1. *Let G be a graph, $E(G) \neq \varnothing$, and G has no leaves. Then, G contains a cycle.*

Proof. Let H be a connected component of G, with $E(H) \neq \varnothing$. If H has no cycle, then H is a tree, and so H contains at least 2 leaves, which is impossible. So H, and thus G, contains a cycle. □

Lemma 6.2. *Let G be a graph in which every vertex has even degree. Then, there exist cycles $C_1, C_2, \ldots, C_k$ in G so that every edge of G belongs to exactly one of them.*

Proof. We induct on $|E(G)|$. The base case is trivial.

By Lemma 6.1, there exists a cycle C_1 of G. Let $H = G \setminus E(C_1)$. By the induction hypothesis, there are cycles $C_2, C_3, \ldots C_k$ in H such that every edge of H belongs to exactly one of them. Then, $C_1, C_2, \ldots, C_k$ is the required list for G. □

Lemma 6.3 (Euler's Theorem)**.** *Let G be a connected graph in which every vertex has even degree in G. Then, there exists a closed trail in G passing through every edge.*

Proof. Choose a closed trail $v_0 e_1 v_1 \ldots e_k v_k$ in G that uses as many edges as possible. Let X be the set of edges not in the walk, and let H be the subgraph of G with $V(H) = V(G)$, $E(H) = X$. The degree of every vertex in H is even. There exists an edge $e \in X$ so that at least one end of it belongs to the walk we selected.

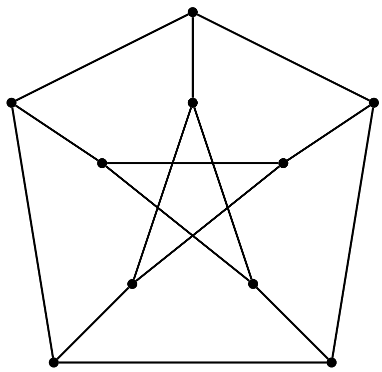
By Lemma 6.2, the edge e belongs to a cycle in H, say C. Then, if v_i is an end of e, tracing our walk until v_i, then following C, and continuing our walk from v_i, yields a trail containing more edges, which is a contradiction, so our walk must have used all edges. □

Definition. A cycles C in a graph G is *Hamiltonian* if $V(G) = V(C)$.

Lemma 6.4. *Let G be a graph. If for some $X \subset V(G)$, $X \neq \varnothing$, $G \setminus X$ has $> |X|$ components, then G does not have a Hamiltonian cycle.*

Proof. Suppose, for a contradiction, that C is a Hamiltonian cycle in G. Let F be the set of edges of C with one end in X and anther in $V(G) - X$. On one hand, $|F| \leq 2|X|$. On the other hand, every component of $G \setminus X$ has at least 2 edges of C with only one end in it. Those edges must be in F. So $|F| \geq 2\text{comp}(G \setminus X)$, and $|X| \geq \text{comp}(G \setminus X)$, contradicting our assumption. □

Remark. The Petersen graph does not have a Hamiltonian cycle, but its vertex set contains no set X which would confirm this from the previous lemma.

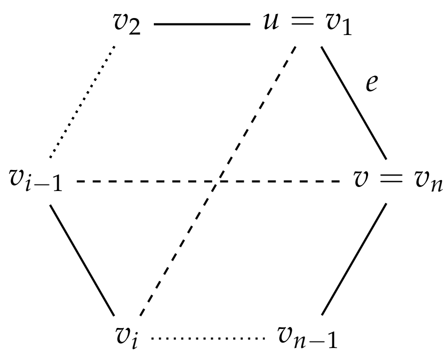


Theorem 6.5 (Dirac-Pósa)**.** *Let G be a simple graph with $|V(G)| \geq 3$. If $\deg(u) + \deg(v) \geq |V(G)|$ for every pair of non-adjacent vertices u and v of G, then G has a Hamiltonian cycle.*

Proof. We proceed by an unusual sort of induction argument.

Fix $n = |V(G)|$. We induct on $\binom{n}{2} - |E(G)|$, which represents the number of absent edges in the graph. For the base case, $|E(G)| = \binom{n}{2}$, so that G is a complete graph, and any cycle of length n in G is Hamiltonian.

Suppose $|E(G)| < \binom{n}{2}$. Let $u, v \in V(G)$ be non-adjacent vertices. Let e be an edge with ends u and v, and let $G' = G + e$. By induction, G' contains a Hamiltonian cycle C. Either C is a Hamiltonian cycle of G, in which case we are done, or $e \in E(C)$. In that case, let $u = v_1, v_2, \ldots, v_n = v$ be the vertices of C in order of traversal.



Let:

$$A = \{v_i : v_{i-1} \text{ adjacent to } v\}$$
$$B = \{v_i : v_i \text{ adjacent to } u\}$$

Then, $|A| = \deg(v), |B| = \deg(u)$, and $A \subset \{v_3, v_4, \ldots, v_n\}$, and $B \subset \{v_2, v_3, \ldots, v_{n-1}\}$, so $A \cup B \subset \{v_2, v_3, \ldots, v_n\}$. So:

$$|A| + |B| = \deg(u) + \deg(v) \geq n > |A \cup B|$$

So $v_i \in A \cap B$ for some i. Then:

$$u, v_2, \ldots, v_{i-1}, v, v_{n-1}, v_{n-2}, \ldots, v_i, u$$

is a Hamiltonian cycle. $\square$

7 Bipartite Graphs

Definition. A *bipartition* of a graph G is a pair of subsets (A, B) of $V(G)$ so that $A \cap B = \varnothing$, $A \cup B = V(G)$, ad every edge of G has one end in A and another in B. A graph is *bipartite* if it admits a bipartition.

Example. The odd cycles C_{2k+1} are not bipartite. The even cycles C_{2k} are bipartite.

Example. Trees are bipartite

Proof. Let T be a tree, and choose $v \in V(T)$. Let $v \in A$. For every other vertex $u \in V(T)$, the unique $u - v$ path in T is either even (i.e. has an even number of edges), in which case let $u \in A$, or odd (i.e. has an odd number of edges), then let $u \in B$. Then, two neighbours always give paths of opposite parity, so (A, B) is a bipartition of T. □

Theorem 7.1. *For every graph G, the following statements are equivalent:*

(1) G is bipartite.

(2) G has no odd closed walk.

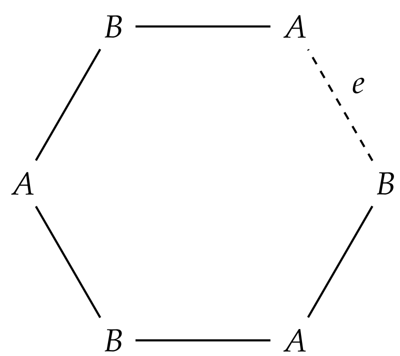
(3) G has no odd cycle.

Proof. (1) ⇒ (2): Vertices on a closed walk must alternatively belong to different parts of the bipartition, so G cannot have an odd closed walk.

(2) ⇒ (3): Clear.

(3) ⇒ (1): We assume WLOG that G is connected. Let T be a spanning tree of G, and let (A, B) be a bipartition of T. It remains to show that every edge $e \in E(G) - E(T)$ has one end in A and another in B.

Let C be a fundamental cycle of e w.r.t. T, and let $P = C \setminus e$. By assumption, C has even length, so P has odd length. Thus, the ends of P, and so the ends of e, are in different parts of the bipartition.



□

Definition. A subgraph H of G is *induced* (or induced by $V(H)$) if H contains all edges of G with both ends in H.

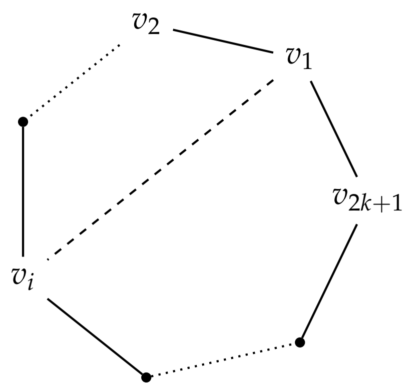
Lemma 7.2. *Let G be a simple graph. Then, the following statements are equivalent:*

(1) G is bipartite.

(2) G contains no induced odd cycle.

Proof. (1) $\Rightarrow$ (2): Follows from Theorem 7.1.

(2) $\Rightarrow$ (1): We will prove the contrapositive, i.e. if G is not bipartite, then G has an induced odd cycle. By Theorem 7.1, G has an odd cycle. Let C be the shortest odd cycle in G, and say C has length $2k+1$. Suppose C is not induced. Then WLOG, $v_1 v_i \in E(G)$ for some $2 \leq i \leq 2k$. Let C_1 be the cycle with vertices $v_1, v_2, \ldots, v_i$, and let C_2 be the cycle with vertices $v_i, v_{i+1}, \ldots, v_{2k+1}, v_1$.



Then, $|V(C_1)| = i < 2k+1 = |V(C)|$, and $|V(C_2)| = 2k+3-i < |V(C)|$, but $|V(C_1)| + |V(C_2)| = 2k+3$, which is odd, so C_1 or C_2 is odd and shorter than C. This is a contradiction, since C was chosen to be the shortest. $\square$

8 Matchings in Bipartite Graphs

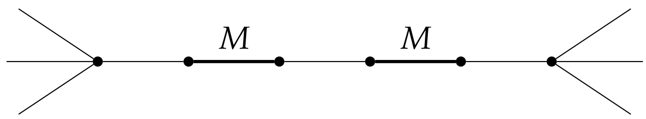
Definition. A *matching* $M \subset E(G)$ in a graph G is a collection of edges so that no edge of M is a loop, and every vertex of G is incident to at most one edge of M.

Definition. Let M be a matching in G. An *M-alternating path* in a path in G, edges of which alternate between belonging to M and $E(G) - M$. Equivalently, every internal vertex of the path is incident with one edge of $M \cap E(P)$ and one edge of $(E(G) - M) \cap E(P)$.

Definition. A path P is M-augmenting if $|P| \geq 1$, it is M-alternating, and the ends of P belong to no edge of M.

Lemma 8.1 (Berge)**.** *Let M be a matching in a graph G. Then, G contains an M-augmenting path if and only if G contains a matching M' with $|M'| > |M|$.*

Proof. Suppose there is an M-augmenting path P. Let $M' = (E(P) - M) \cup (M - E(P))$. Then, $|M'| = |M| + 1$.



Conversely, consider $M' \cup M$. Let $H \leq G$ be defined by $V(H) = V(G)$, and $E(H) = M' \cup M$. Note that $\deg_H(v) \leq 2$ for all $v \in V(H)$, and there are no loops in H. By case analysis, one can show that every component of G is a path or a cycle. There exists a component C of H so that $|E(C) \cap M'| > |E(C) \cap M|$. If C is a cycle, then we have $|E(C) \cap M'| = |E(C) \cap M|$. So C must be a path. Then, C is both M-alternating and M'-alternating. The only way to satisfy the inequality is if the first and last edges of C are in M', so C is M-augmenting. Since C is a component, then the ends belongs to no edge of M. □

Theorem 8.2 (König)**.** *Let G be a bipartite graph. Then, the following are equivalent:*

(1) G has a matching M with $|M| \geq k$.

(2) There does not exist $X \subset V(G)$ with $|X| < k$ such that every edge of G has an end in X.

Proof. (1) $\Rightarrow$ (2): Suppose for a contradiction that there exists a set as in (2). Then, every edge of M must have an end at some vertex of X, and the corresponding vertices of X are distinct for distinct edges of M. So $|X| \geq |M|$, which is a contradiction.

(2) $\Rightarrow$ (1): Let M be a matching of G of maximal size. Also, let (A, B) be a bipartition of G. Let $A' \subset A$ be the set of vertices of A not incident to edges to M. Let $B' \subset B$ be defined similarly. Let Z be the set of vertices of G so that there is an M-alternating path with one end in some $v \in A'$ and another in Z. By Lemma 8.1, there exists no M-augmenting path and so no M-alternating path can start in A' and end in B'. So B' does not intersect Z. We thus list 3 properties:

a) $B' \cap Z = \varnothing$.

b) If an edge of M' has an end in Z, then it has both ends in Z.

c) Every edge of G with one end in $A \cap Z$ has another in $B \cap Z$, because an M-alternating path traced from $v \in A'$ uses edges of M from B to A, and edges of $E(G) - M$ from A to B.

Let $X = (A - Z) \cup (B \cap Z)$. Then, By a) and b), X contains exactly one end of every edge of M, and so $|X| = |M|$. Moreover, every edge of G has an edge in X by c). From (2), $|X| \geq k$, so $|M| \geq k$. □

Definition. We let $\nu(G)$, the *matching number* of G, denote the maximum size of a matching in G.

Definition. A subset $X \subset V(G)$ is a *vertex cover* of G if every edge of G has an end in X.

Definition. We let $\tau(G)$, the *vertex cover number* of G, denote the minimum size of a vertex cover in G.

Corollary (König)**.** *If G is bipartite, then $\nu(G) = \tau(G)$.*

Remark.

- $\nu(G) \leq \tau(G)$ for any graph G.
- $\nu(C_{2k+1}) = k$, $\tau(C_{2k+1}) = k + 1$.
- $\nu(K_n) = \lfloor n/2 \rfloor$, $\tau(K_n) = n - 1$.
- The vertex set of any maximal matching is a vertex cover, so:

$$\frac{\tau(G)}{2} \leq \nu(G) \leq \tau(G)$$

Definition. A matching M *covers* $Y \subset V(G)$ if every vertex in Y is incident to an edge of M. M is *perfect* if it covers $V(G)$.

Corollary 8.3. *Let G be bipartite, with every vertex of the same degree $d > 0$. Then, G has a perfect matching.*

Proof. We want to show that $\nu(G) \geq \frac{|V(G)|}{2}$. By Theorem 8.2, it suffices to show that $|X| \geq \frac{|V(G)|}{2}$ for any vertex cover X.

Now, $2|E(G)| = \sum \deg(v)$, so $|E(G)| = \frac{d|V(G)|}{2}$. X contains ends of at most $d|X|$ edges. Thus, if X is a vertex cover, then:

$$d|X| \geq |E(G)| = \frac{d|V(G)|}{2} \Rightarrow |X| \geq \frac{|V(G)|}{2}$$

□

Theorem 8.4 (Hall)**.** *Let G be bipartite, with bipartition (A, B). Then, the following are equivalent:*

(1) There exists a matching in G covering A.

(2) For all $Y \subset A$, vertices in Y have at least $|Y|$ neighbours in B.

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (1): We want to show that $\nu(G) \geq |A|$, so we will show that $\tau(G) \geq |A|$ by Theorem 8.2. Let X be a vertex cover in G. Let $Y = A - X$ By assumption, $|B \cap X| \geq |Y| = |A - X|$. Thus:

$$|X| = |A \cap X| + |B \cap X| \geq |A \cap X| + |A - X| = A$$

□

9 Menger's Theorem and Separations

Definition. A pair (A, B), $A, B \subset V(G)$ is called a *separation* if $A \cup B = V(G)$, and no edge of G has one end in $A - B$ and another in $B - A$. In other words, any edge from A to B has an end in $A \cap B$.

The *order* of a separation (A, B) is $|A \cap B|$.

Theorem 9.1. *For $Q, R \subset V(G)$, $k \in \mathbb{Z}^+$, then exactly one of the following holds:*

(1) There exist paths $P_1, P_2, \ldots, P_k$, pairwise vertex disjoint (i.e. $V(P_i) \cap V(P_j) = \varnothing$ for $i \neq j$) and each P_i has one end in Q and another in R.

(2) There exists a separation (A, B) of G of order $< k$ such that $Q \subset A$ and $R \subset B$.

Proof. We need to show that either (1) or (2) holds. We will induct on $|V(G)| + |E(G)|$. The base case is clear. Let us first consider three cases where we can finish the proof; these cases are captured in the following three claims:

Claim 9.1.1. *If there exists a separation (A', B') with $Q \subset A'$, $R \subset B'$, of order k, and $A' \neq V(G)$, $B' \neq V(G)$, then the theorem holds.*

Proof. Apply the induction hypothesis to G' which is a subgraph of G induced on A', with the sets Q' and $A' \cap B'$. Either (1) or (2) holds.

Suppose first that (2) holds, and there is a separation (A'', B'') of G' with $Q \subset A''$, $A' \cap B' \subset B''$, of order $< k$. Consider a separation $(A'', B'' \cup B')$. This is a separation of G of order $< k$, so (2) holds for G.

Suppose that (1) holds. There are paths $P'_1, P'_2, \ldots, P'_k$ pairwise vertex disjoint each with one end in Q and another in $A' \cap B'$. Apply the induction hypothesis to the graph induced on B' with sets $A' \cap B'$ and R. Again, we may suppose that (1) holds. So there are paths $P''_1, P''_2, \ldots, P''_k$, pairwise vertex disjoint, each with one end in $A' \cap B'$ and another in R. We may assume P'_i shares an end with P''_i for each $1 \leq i \leq k$. Then, paths $P'_1 \cup P''_1, P'_2 \cup P''_2, \ldots, P'_k \cup P''_k$ show that (1) holds for G. □

Claim 9.1.2. *If $Q \cap R \neq \varnothing$, then the theorem holds.*

Proof. If $v \in Q \cap R$, then apply the induction hypothesis to $G - v$, $Q - V$, $R - v$, and $k - 1$. □

Claim 9.1.3. *If $k = 1$, then the theorem holds.*

Proof. If no path has an end in Q and an end in R, then taking A to be the union of components containing a vertex of Q, and $B = V(G) - A$, we get that (2) holds. $\square$

Now we are ready to finish the proof of Theorem 9.1. Consider any edge $e \in E(G)$ with ends u and v, and apply the induction hypothesis to $G \setminus e$. If (1) holds, we are done. So assume (2) holds for $G \setminus e$. We may assume $u \in A - B$ and $v \in B - A$, as otherwise (2) holds. Consider now separations $(A \cup \{v\}, B)$ and $(A, B \cup \{u\})$. These separations are of order at most k, so the theorem holds by Claim 9.1.1 unless we have both $A \cup \{v\} = V(G)$ and $B \cup \{u\} = V(G)$.

Suppose this is the case. Then $|A \cap B| < k$, and $|V(G)| \leq k + 1$, and $|Q \cap R| = 0$ by Claim 9.1.2. $|Q| \geq k$, $|R| \geq k$, so $|V| \geq |Q \cup R| \geq 2k$. However, $2k > k + 1$ by Claim 9.1.3; a contradiction. $\square$

Remark. Theorem 9.1 $\Rightarrow$ Theorem 8.2 (König's theorem). Indeed, let G be a bipartite graph with bipartition (Q, R). Existence of a matching of size k is equivalent to the existence of k pairwise disjoint paths from Q to R. For a separation (A, B), with $Q \subset A$, $R \subset B$, we have $|A \cap B| = |A \cap R| + |B \cap Q|$. Let $X = (B \cap Q) \cup (A \cap R) = A \cap B$. Then X is a vertex cover.

Theorem 9.2 (Menger). *Let $s, t \in V(G)$ be non-adjacent, $k \in \mathbb{Z}^+$. Then, exactly one of the following holds:*

(1) There exist paths $P_1, P_2, \ldots, P_k$, pairwise vertex disjoint except at s and t, with one end in s and another in t.

(2) There exists a separation (A, B) of G of order $< k$ with $s \in A - B$ and $t \in B - A$.

Proof. Both cannot hold. To show that at least (1) or (2) holds, let Q be the set of vertices adjacent to s, and R be the set of vertices adjacent to t, and $G' = G \setminus \{s, t\}$. Apply Theorem 9.1 to G', Q, and R. If (1) holds, then paths sP_it are corresponding paths which satisfy this theorem. If (2) holds, then let (A', B') be the corresponding separation of G'. Let $A = A' \cup \{s\}$, and $B = B' \cup \{t\}$. Then, (A, B) is a separation of G showing that (2) holds. $\square$

Definition. A graph is *k-connected* if $|V(G)| \geq k + 1$ and $G \setminus X$ is connected for every $X \subset V(G)$ with $|X| < k$.

Corollary 9.3. *Let G be a k-connected graph, $s, t \in V(G)$, $s \neq t$. Then, there exist paths $P_1, P_2, \ldots, P_k$ in G, each with one end in s and another in t, and otherwise pairwise vertex disjoint.*

Proof. If s and t are non-adjacent, then we can apply Menger's theorem. If s and t are adjacent, then let G' be the graph obtained from G be deleting all edges from s to t. Then, G' is $(k-1)$-connected, so we can apply this corollary to obtain paths $P_1, P_2, \ldots, P_{k-1}$, and let P_k be a one edge $s-t$ path. $\square$

Definition. For $X \subset V(G)$, let $\delta(X)$ denote the set of all edges of G with one end in X and another in $V(G) - X$.

Definition. The *line graph* $L(G)$ of G has $V(L(G)) = E(G)$, and two vertices of $L(G)$ are adjacent if and only if the edges they represent share an end in G.

Theorem 9.4. *Let $s, t \in V(G)$ be distinct, $k \in \mathbb{Z}^+$. Then, exactly one of the following holds:*

(1) There exist paths $P_1, P_2, \ldots, P_k$ from s to t, pairwise edge disjoint.

(2) There is a set of vertices $X \subset V(G)$ with $s \in X$, $t \in V(G) - X$, and $|\delta(X)| < k$.

Proof. Let H be the line graph of G. It is easy to check that a path in the line graph contains an edge set of a path in the original graph, with the first edge e and last edge f.

Now, both (1) and (2) cannot hold. We show that at least one does. Let $Q \subset E(G)$ be the set of edges incident to s, $R \subset E(G)$ be the set of edges incident to t. Apply Theorem 9.1 to H, Q, and R. Either (1) holds in 9.1, and then (1) holds in this theorem by the above observation. Otherwise, (2) holds in 9.1, so there are $A, B \subset E(G)$ with $Q \subset A$, $R \subset B$, and $|A \cap B| < k$, $A \cup B = E(G)$, and no edge of $A - B$ shares an end with an edge in $B - A$. Let X be the set of vertices only incident to edges of A, except possibly for $t \in V(G) - X$. Then, one can check that X satisfies (2). $\square$

10 Directed Graphs and Network Flows

Definition. A *directed graph* (or digraph) is a graph where for every edge, one of its ends is chosen as a *head* and the other as a *tail*. An edge is said to be *directed* from its tail to its head.

Definition. A *directed path* from s to t in a directed graph is a path in which every edge is traversed from its tail to its head as we follow the path from s to t.

Definition. For a set $X \subset V(G)$, let $\delta^+(X)$ denote the set of all edges with tail in X and head in $V(G) - X$, and let $\delta^-(X) = \delta^+(V(G) - X)$.

Lemma 10.1. *Let G be a digraph. Let $s, t \in V(G)$. Then, exactly one of the following holds:*

(1) There exists a directed path in G from s to t.

(2) There exists $X \subset V(G)$, $s \in X$, $t \notin X$, and $\delta^+(X) = \varnothing$.

Proof. (1) and (2) cannot hold simultaneously: the preceding the first vertex of $V(G) - X$ along a directed $s - t$ path must lie in $\delta^+(X)$.

In addition, at least one (1) or (2) must hold: let X be the set of vertices $v \in V(G)$ so that there is a directed path from s to v. If $t \in X$, then (1) holds. If $t \notin X$, then (2) holds. □

Definition. Let F be a digraph, $s, t \in V(G)$. A function $\phi : E(G) \to \mathbb{R}^+$ is an $s - t$ *flow* if:

$$\sum_{e \in \delta^-(v)} \phi(e) = \sum_{e \in \delta^+(v)} \phi(e)$$

for every $v \in V(G) - \{s, t\}$.

Definition. The *value of* ϕ is $\sum_{e \in \delta^+(s)} \phi(e) - \sum_{e \in \delta^-(s)} \phi(e)$.

Lemma 10.2. *Let G be a digraph, ϕ an $s - t$ flow of value k. Then, for all $X \subset V(G)$ such that $s \in X$, $t \notin X$ then:*

$$\sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) = k$$

Proof. We first note that:

$$k = \sum_{v \in X} \left(\sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e) \right) = \sum_{e \in E(G)} \phi(e)(n_1(e) - n_2(e))$$

Where:

$$n_1(e) = \begin{cases} 1 & \text{if } e \text{ has a tail in } X; \\ 0 & \text{otherwise.} \end{cases}$$

$$n_2(e) = \begin{cases} 1 & \text{if } e \text{ has a head in } X; \\ 0 & \text{otherwise.} \end{cases}$$

So:

$$k = \sum_{e \in E(G)} \phi(e)(n_1(e) - n_2(e)) = \sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e)$$

□

Lemma 10.3. *Let ϕ be an integral $s - t$ flow of value k. Then there exist directed paths $P_1, P_2, \dots, P_k$ from s to t in G, and every edge e of G belongs to at most $\phi(e)$ paths.*

Proof. We proceed by induction on k. The base case is trivial.

Suppose $k \geq 1$. Let G' be a digraph with $V(G') = V(G)$ and $E(G')$ consisting of edges $e \in E(G)$ with $\phi(e) \geq 1$. By Lemma 10.1, either G' has a directed $s - t$ path P_k, or there exists $X \subset V(G)$ such that $\phi(e) = 0$ for all $e \in \delta^+(X)$, and $s \in X$, $t \notin X$. The second outcome contradicts Lemma 10.2. Let ϕ' be defined as follows:

$$\phi'(e) = \begin{cases} \phi(e) & \text{if } e \notin E(P_k); \\ \phi(e) - 1 & \text{if } e \in E(P_k). \end{cases}$$

The value of ϕ' is $k - 1$. Applying the induction hypothesis to ϕ', we obtain paths $P_1, P_2, \dots, P_{k-1}$ directed from s to t, so that every edge e is in at most $\phi'(e)$ of those paths. The paths $P_1, P_2, \dots, P_{k-1}, P_k$ satisfy the requirements of the theorem. □

Definition. Let G be a digraph, s, t be distinct vertices of G. For every $e \in E(G)$, let $c(e) \in \mathbb{Z}^+$ be the *capacity* of this edge. And $s - t$ flow ϕ is *c-admissible* if $\phi(e) \leq c(e)$ for all $e \in E(G)$.

Definition. A path P be with end in s and another in some vertex v is called an *augmenting path for ϕ* if:

(1) $\phi(e) \leq c(e) - 1$ for every edge e which is used in the forward direction as P is traversed from s to v.

(2) $\phi(e) \geq 1$ if $e \in E(P)$ is traversed in the opposite direction.

Lemma 10.4. *Let G, s, t, c be as above. Let ϕ be an integral c-admissible $s - t$ flow. If there exists an augmenting path P for ϕ from s to t, then there is a c-admissible $s - t$ flow of value larger than the value of ϕ.*

Proof. Let ψ be defined as follows:

$$\psi(e) = \begin{cases} 1 & \text{if } P \text{ traverses } e \text{ in the forward direction;} \\ -1 & \text{if } -P \text{ traverses } e \text{ in the backwards direction;} \\ 0 & \text{otherwise.} \end{cases}$$

ψ is an integral $s - t$ flow of value 1. In addition, $(\phi + \psi)$ is an integral $s - t$ flow of value 1 lager than the value of ϕ, and $(\phi + \psi)$ is c-admissible. □

Theorem 10.5 (Ford-Fulkerson, Max Flow-Min Cut)**.** *Let G, s, t, c be as above, $k \in \mathbb{Z}^+$. Then, exactly one of the following holds:*

(1) There exists a c-admissible $s - t$ flow on G of value k.

(2) There exists $X \subset V(G)$, $s \in X$, $t \notin X$, so that:

$$\sum_{e \in \delta^+(X)} c(e) < k$$

Proof. Let us first show that (1) and (2) cannot both hold: let ϕ be a c-admissible $s - t$ flow of value k. By Lemma 10.2:

$$k = \sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) \leq \sum_{e \in \delta^+(X)} c(e) + 0$$

which is a contradiction.

Let ϕ be a c-admissible $s - t$ flow of maximal value. We may assume that the value of ϕ is $< k$. Let X be the set of all vertices $v \in V(G)$ so that there exists an augmenting path for ϕ from s to v. We show that X satisfies (2).

Indeed, $s \in X$, $t \notin X$, by Lemma 10.4. $\phi(e) = c(e)$ for all $e \in \delta^+(X)$, otherwise an augmenting path to the end of e in X can be extended along e. Also, $\phi(e) = 0$ for all $e \in \delta^-(X)$. By Lemma 10.2, the value of ϕ (which is less than k), is:

$$\sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) = \sum_{e \in \delta^+(X)} c(e) - 0 < k$$

□

Remark. Theorem 10.5 ⇒ Theorem 9.2 (Menger's theorem)

11 Independent Sets and Gallai's Equations

Definition. An *independent set* $X \subset V(G)$ is a set so that no edge of G has both ends in X (in particular, no loop is incident to a vertex of X).

Definition. We let $\alpha(G)$, the independence number of G, denote the maximum size of an independent set of G.

Definition. An *edge cover* $L \subset E(G)$ is a set so that every vertex of G is incident to an edge of L.

Definition. We let $\rho(G)$, the *edge covering number* of G, denote the minimum size of an edge cover of G.

Remark.

- $\alpha(G) \leq \rho(G)$ and $\rho(G) \geq \frac{|V(G)|}{2}$.
- $\alpha(C_{2k+1}) = k$, $\rho(C_{2k+1}) = k+1$.
- $\alpha(K_n) = 1$, $\rho(K_n) = \lceil n/2 \rceil$.

Lemma 11.1. *For any graph G:*

$$\alpha(G) + \tau(G) = |V(G)|$$

Proof. Let X be an independent set of G. Then, $V(G) - X$ is a vertex cover. If X is an independent set of G of maximum size, then $|V(G)| - \alpha(G) = |V(G) - X| \geq \tau(G)$, so $\alpha(G) + \tau(G) \leq |V(G)|$. One similarly shows the opposite inequality. □

Theorem 11.2 (Gallai)**.** *Let G be a connected simple graph with $|V(G)| \geq 2$. Then:*

$$\rho(G) + \nu(G) = |V(G)|$$

Proof. First, we show that $\nu(G) + \rho(G) \leq |V(G)|$: let M be a matching in G with $|M| = \nu(G)$. Let L be obtained from M by adding one extra edge incident to every vertex not covered by M. L is an edge cover, and:

$$|L| = |M| + (|V(G)| - 2|M|) = |V(G)| - |M| = |V(G)| - \nu(G)$$

It remains to show that $\rho(G) + \nu(G) \geq |V(G)|$: let H be the graph with $V(G) = V(H)$, $E(H) = L$. Because L is minimal, every edge of H has an end of degree 1. H is a forest, and every component of H contains an edge. Let M be a matching obtained by taking a single edge from every component of H:

$$|M| = \text{comp}(H) = |V(H)| - |E(H)| = |V(G)| - |L| = |V(G)| - \rho(G)$$

□

Corollary 11.3. *Let G be a connected, simple, bipartite graph with $|V(G)| \geq 2$. Then, $\alpha(G) = \rho(G)$.*

Proof. The proof is immediate from König's theorem and the previous two results. □

12 Ramsey's Theorem

This section is based on the philosophy of Ramsey theory: in any large system, however chaotic or complicated, there is a subsystem with a special structure.

Example. In every group of 6 people, there are 3 such that any 2 of these are friends or there are 3 such that no 2 of these are friends.

Definition. A *clique* in a graph G is a subset $X \subset V(G)$ such that every 2 vertices in X are adjacent. In other words, a clique is a complete subgraph.

Definition. Let $R(s,t)$ denote the minimum number n such that every (simple) graph G on n vertices either contains an independent set of size s or a clique of size t. $R(s,t)$ is called a *Ramsey number*.

Theorem 12.1 (Ramsey, Erdős-Szekeres). *Let $s,t \geq 1$ be integers. Then $R(s,t)$ exists. Moreover, for $s,t \geq 2$:*

$$R(s,t) \leq R(s-1,t) + R(s,t-1)$$

Proof. We prove the theorem, and in particular the inequality, by induction on $s+t$. First observe that $R(1,t) = R(s,1) = 1$, so the base case holds.

We need to show that if $s,t \geq 2$, and $R(s-1,t), R(s,t-1)$ exist, then every graph on $n = R(s-1,t) + R(s,t-1)$ vertices has an independent set of size s and a clique of size t. Let G be a graph with $|V(G)| = n$. Let $v \in V(G)$.

If $\deg(v) \geq R(s,t-1)$, consider the neighburs of v. Among them, by definition of $R(s,t-1)$, there is either an independent set on s vertices, or a clique of size $t-1$. In the second case, we can add v to this clique to obtain one of size t.

If $\deg(v) \leq R(s,t-1) - 1$, then v has at least:

$$n - (R(s,t-1)-1) - 1 = n - R(s,t-1) = R(s-1,t)$$

non-neighbours. Repeat the previous argument above on non-neighbours of v to complete the proof. □

Corollary 12.2. *For $s,t \geq 1$, we have:*

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

Proof. By induction on $s+t$. When $s=1$ or $t=1$:

$$R(s,t) = \binom{s-1}{s-1} = \binom{t-1}{0} = 1$$

By the induction hypothesis:

$$R(s-1,t) \leq \binom{(s-1)+t-2}{(s-1)-1}$$
$$R(s,t-1) \leq \binom{s+(t-1)-2}{s-1}$$

So by Theorem 12.1:

$$R(s,t) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$$

by Pascal's identity. □

Remark. This last corollary implies that $R(3,3) \leq 6$. In fact, $R(3,3) = 6$, since C_5 contains no clique of size 3 and no independent set of size 3. This confirms the truth of the example presented at the beginning of this section.

Definition. Let $R_k(s_1, s_2, \ldots, s_k)$, a *multicolour Ramsey number*, be the minimum integer n so that in every colouring of edges of K_n with colours $\{1, 2, \ldots, k\}$, for some i there exists a complete subgraph on s_i vertices with all edges of colour i.

Theorem 12.3. *The multicolour Ramsey number $R_k(s_1, s_2, \ldots, s_k)$ exists for all positive integers $s_1, s_2, \ldots, s_k$.*

Proof. We induct on k. When $k=1$ and $k=2$, the theorem is known to hold by Theorem 12.1. Otherwise:

$$R_k(s_1, s_2, \ldots, s_k) \leq R_{k-1}(R_2(s_1, s_2), s_3, \ldots, s_k)$$

□

Theorem 12.4 (Schur). *For every $k \geq 1$, there exists n so that in every colouring of $\{1, 2, \ldots, n\}$ in k colours, one can find a monochromatic solution to $x + y = z$.*

Proof. Let $n = R_k(3,3,\ldots,3)$. Let $G = K_n$ with vertex set $\{1,2,\ldots,n\}$. For $x, y \in \{1,2,\ldots,n\}$, coloour the edge joining x and y by the colour of $|x-y|$. Now, by the choice of n, there exist $a,b,c \in V(G)$ such that $|a-b|, |a-c|, |b-c|$ all have the same colour. WLOG, let $a < b < c$. Let $x = b-a, y = c-b, z = c-a$. We have $x+y=z$, and they all have the same colour. □

Theorem 12.5 (Schur)**.** *For every $m \geq 1$, there exists p_0 such that for every prime $p > p_0$, the modular equation $x^m + y^m \equiv z^m (mod\ p)$ has a nontrivial solution. In other words, Fermat's last theorem fails in prime fields of sufficiently large order.*

Proof. Let $\mathbb{Z}_p^\times = \langle g \rangle$. Colour $\{1,2,\ldots,p-1\}$ in m colours as follows: if $x \in \{1,2,\ldots,p-1\}$ is such that $x \equiv g^{i_x m + j_x}$, $0 \leq j_x \leq m-1$, then colour x in the colour j_x. By Theorem 12.4, for $p > p_0$ for some p_0, there exist $x,y,z \in \{1,2,\ldots,p-1\}$ such that $x+y=z$ and x,y,z are monochromatic:

$$x \equiv g^{mi_x + r}, y \equiv g^{mi_y + r}, z \equiv g^{mi_z + r}$$

So:

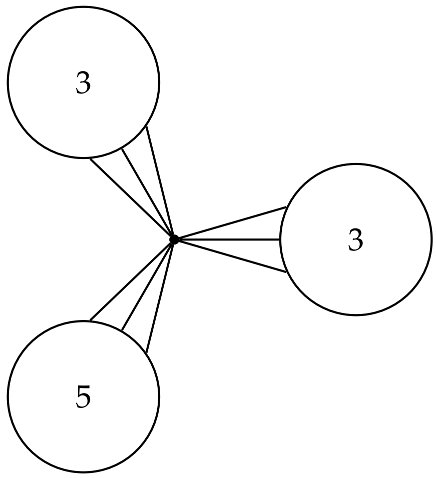
$$g^{mi_x + r} + g^{mi_y + r} \equiv g^{mi_z + r} \Rightarrow (g^{i_x})^m + (g^{i_y})^m \equiv (g^{i_z})^m$$

is a nontrivial solution. □

13 Matchings and Tutte's Theorem

Remark.

- A matching M is perfect if and only if $|M| = \frac{|V(G)|}{2}$.
- A graph has no perfect matching if $|V(G)|$ is odd.
- Another example of a graph with no perfect matching:



Definition. For a graph H, let $\text{comp}_o(H)$ denote the number of components of H with odd numbers of vertices.

Theorem 13.1 (Tutte)**.** *A graph G has a perfect matching if and only if:*

$$comp_o(G \setminus X) \leq |X|$$

for all $X \subset V(G)$.

Proof. If $\text{comp}_o(G \setminus X) > |X|$, we show that there is no perfect matching. In a perfect matching, for every odd component of $G \setminus X$, some matching edge joins this component to X. There are not enough vertices of X for this to be possible.

Conversely, we proceed by lengthy induction on $|V(G)|$. The base case is, of course, trivial.

Claim 13.1.1. *$|V(G)|$ is even, since $comp_o(G) \leq 0$ by assumption.*

Claim 13.1.2. *We say that X is* critical *if $comp_o(G \setminus X) \geq |X| - 1$. If X is critical, then $|X| = comp_o(G \setminus X)$.*

Proof. Note that $|V(G)| \equiv |X| + \text{comp}_o(G \setminus X) (\text{mod } 2)$. By Claim 13.1.1, $|X| \equiv \text{comp}_o(G \setminus X) (\text{mod } 2)$, so $|X| = \text{comp}_o(X) + 1$ is impossible. □

Claim 13.1.3. *There exists a critical set, say $X = \varnothing$.*

Let X be a maximal critical set in G, and $k = |X|$. By Claim 13.1.2, let $C_1, C_2, \ldots, C_k$ be odd components of $G \setminus X$.

Claim 13.1.4. *There are no even components in $G \setminus X$.*

Proof. Suppose $v \in G \setminus X$ belongs to an even component. Then $\text{comp}_o(X \cup \{v\}) \geq |X|$, and $|X \cup \{v\}| = |X| + 1$, and $X \cup \{v\}$ is critical, contradicting the maximality of X. □

Claim 13.1.5. *For all i, $v \in V(C_i)$, $C_i \setminus \{v\}$ has a perfect matching.*

Proof. Suppose not. Then, by the induction hypothesis, Tutte's condition is violated, so there exists $Y \subset V(C_i) \setminus \{v\}$ such that $C_i \setminus Y \setminus \{v\}$ has $> |Y|$ odd components. Then:

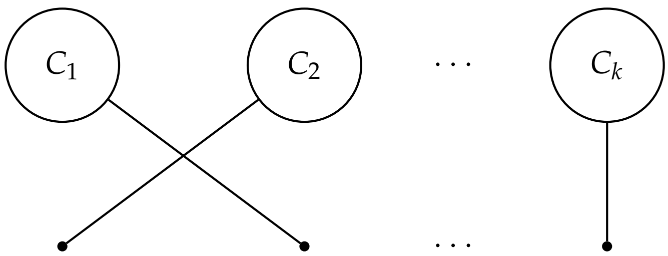
$$\text{comp}_o(X \cup Y \cup \{v\}) \geq k - 1 + |Y| + 1 = |X| + |Y| = |X \cup Y \cup \{v\}| - 1$$

so $X \cup Y \cup \{v\}$ is critical, which is a contradiction. □

From the past 5 claims, we obtain that if $|X| = k$, then $G \setminus X$ consists of k odd components $C_1, C_2, \ldots, C_k$, so that for each $v \in V(C_i)$, $C_i \setminus \{v\}$ has a perfect matching.

Claim 13.1.6. *There exists a matching M' in G covering X such that every C_i contains an end of exactly one edge of M'.*

Proof. Let H be an auxiliary bipartite graph with bipartition (C, X), where $C = \{C_1, C_2, \ldots, C_k\}$, and C_i is joined by a edge to $x \in X$ if x has a neighbour in C_i.



If H has a perfect matching, then claim 6 holds. By Theorem 9.4, H has a matching covering C unless for some $Z \subset C$, there are fewer than $|Z|$ vertices in X with neighbours in Z. Let $Y \subset X$ be vertices of X with neighbours in Z. Every odd component in Z is also an odd component of $G \setminus Y$, so $\text{comp}_o(G \setminus Y) > |Y|$, which contradicts the theorem. $\square$

Let v_i be an end of an edge of M' in C_i for $1 \leq i \leq k$. By Claim 13.1.5, $C_i \setminus v_i$ has a perfect matching M_i. We get that $M' \cup M_1 \cup M_2 \cup \ldots \cup M_k$ is a perfect matching of G, as desired. $\square$

Theorem 13.2 (Tutte-Berge)**.** *A graph G has a matching of size k if and only if:*

$$comp_o(G \setminus X) \leq |X| + |V(G)| - 2k$$

for every $X \subset V(G)$.

Proof. We construct a graph G' from G by adding $|V(G)| - 2k$ vertices and joining them to every other vertex (in particular, to each other). G' has a perfect matching if and only if G has a matching of size k: if M is a matching of G of size k, then matching the uncovered vertices of G with vertices in $V(G') - V(G)$ creates a perfect matching of G'. On the other hand, in a perfect matching on G', $2k$ vertices of G must be matched to each other. G' has a perfect matching if and only if:

$$\text{comp}_o(G' \setminus Y) \geq |Y|$$

for all $Y \subset V(G')$. If $V(G') - V(G) \not\subset Y$, then $\text{comp}_o(G' \setminus Y) \leq 1$. In this case, Tutte's condition holds on G', unless $Y = \varnothing$ and $|V(G')|$ is odd, but $|V(G')| = 2|V(G)| = 2k$.

So we only consider Y such that $V(G') - V(G) \subset Y$, and let $X = Y \cap V(G)$. Then, $|Y| = |X| + |V(G)| - 2k$, and $\text{comp}_o(G' \setminus Y) = \text{comp}_o(G \setminus X)$, because $G' \setminus Y = G \setminus X$. So Tutte's condition on G' is equivalent to the inequality in the theorem. $\square$

Definition. If G is a graph in which every vertex has degree k, then G is said to be k-regular.

Theorem 13.3. *Let G be a 3-regular graph. If G has no cut edge, then G has a perfect matching.*

Proof. By Theorem 13.1, it is sufficient to check that $\text{comp}_o(G \setminus X) \leq |X|$ for all $X \subset V(G)$.

WLOG, G is connected. Let $X \subset V(G)$. Let $C_1, C_2, \ldots, C_k$ be odd components of $G \setminus X$. Let F_i be the set of edges between $V(C_i)$ and X. Then, $|F_i| \geq 1$, and $|F_i| \neq 1$, since there are no cut edges. In fact, $|F_i|$ is odd, since:

$$3|C_i| = \sum_{v \in V(C_i)} \deg(v) = 2|E(C_i)| + |F_i|$$

So $|C_i| \equiv |F_i| \pmod 2$, and $|F_i|$ is odd. So $|F_i| \geq 3$, so there are at least $3k$ edges between $\bigcup_{i=1}^{k} C_i$ and X. $\square$

14 Vertex Colouring

Definition. Let G be a graph, S a set of size k. The function $\varphi : V(G) \to S$ is called a *(proper) k-colouring* if for all $e \in E(G)$ with ends u and v, $\varphi(u) \neq \varphi(v)$. Elements of S are called *colours*. The set of all vertices of the same colour is called a *colour class*.

Definition. The *chromatic number* $\chi(G)$ is the minimum k such that G admits a k-coloring. If G has a loop, then $\chi(G)$ is not well-defined, and we sometimes say that $\chi(G) = \infty$.

Remark. A graph is 2-colourable if and only if it is bipartite. In an algorithmic sense, it is "easy" to check if a graph is bipartite (or 2-colourable) using a colouring of a spanning tree. It is "hard" to check if a graph is 3-colourable: the best known algorithms take exponential time.

Definition. We let $\omega(G)$, the *clique number* of G, denote the size of the largest clique in G.

Lemma 14.1. *Let G be a loopless graph. Then:*

(1) $\chi(G) \geq \omega(G)$.

(2) $\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil$.

Proof. Vertices of a complete subgraph in G must receive pairwise distinct colours in any colouring. So (1) follows.

If $V_1, V_2, \ldots, V_k$ are colour classes in a k-colouring of G, where $k = \chi(G)$, then $|V_i| \leq \alpha(G)$ for all i, so:

$$|V(G)| = \sum_{i=1}^{\chi(G)} |V_i| \leq \chi(G)\alpha(G)$$

and (2) follows. □

Greedy Colouring Algorithm

Let $(v_1, v_2, \ldots, v_n)$ be an ordering of all vertices of a graph G. We present a greedy algorithm for vertex colouring G w.r.t. this ordering:

input: A graph G and a vertex enumeration $(v_1, v_2, \ldots, v_n)$.

algorithm:

- Colour v_1 in colour 1.
- Suppose that $\{v_1, v_2, \ldots, v_{i-1}\}$ have been coloured. Let the colour of v_i be the smallest possible positive integer that is still a valid colour for v_i.
- Repeat this process until all vertices are coloured.

output: A vertex colouring for G.

Definition. Let $\Delta(G)$ denote the maximum degree of any vertex in a graph G.

Definition. A graph G is called *k-degenerate* if every subgraph of G has a vertex of degree at most k.

Remark. Every graph is $\Delta(G)$-degenerate. Also, a graph is 1-degenerate if and only if it is a forest.

Lemma 14.2. *If G is a loopless k-degenerate graph, then $\chi(G) \leq k+1$. In particular, $\chi(G) \leq \Delta(G)+1$ for loopless graphs.*

Proof. We will create an ordering $(v_1, v_2, \ldots, v_n)$ of $V(G)$ which will produce an appropriate vertex colouring from the greedy colouring algorithm. We begin by listing the vertices of high index. Let $v_n \in V(G)$ be such that $\deg(v_n) \leq k$. Such a vertex exists, since G is k-degenerate. Suppose then, that numbers were assigned to vertices $\{v_n, v_{n-1}, \ldots, v_{n-i}\} = U$. Let $G' = G \setminus U$. Let v_{n-i-1} be the vertex of G' with degree in G' at most k. Continue this process until all vertices are numbered.

The greedy algorithm w.r.t. this ordering requires at most $k+1$ colours: suppose $\{v_1, v_2, \ldots, v_{i-1}\}$ are coloured. Then, v_i has at most k neighbours among them, so one of the colours $\{1, 2, \ldots, k+1\}$ is not used by neighbours of v_i and can be used for v_i. □

Remark. There are graphs which require $\chi(G) = \Delta(G)+1$ colours, for instance, complete graphs and odd cycles.

Theorem 14.3 (Brooks)**.** *Let G be a connected loopless graph which is not complete and not an odd cycle. Then, $\chi(G) \leq \Delta(G)$.*

Proof. This is the second lengthy proof of this course. Get ready.

Claim 14.3.1. *Let H be a connected graph. Let $v \in V(H)$. Then, there exists an order $(v_1, v_2, \ldots, v_{k-1}, v)$ such that for every vertex $v_i \in V(H) \setminus \{v\}$, some neighbour of v_i appears after it in the ordering.*

Proof. Let T be a spanning tree of H. Order vertices of H in non-increasing order of distance to v. $\square$

Suppose Brooks' theorem is false. Choose G for which it fails with $|V(G)|$ minimum.

Claim 14.3.2. *It must be that* $\Delta(G) > 2$.

Proof. If $\Delta(G) = 1$, then G is complete. If $\Delta(G) = 2$, then either G is a path or a cycle and so bipartite, unless G is an odd cycle. So the theorem holds if $\Delta(G) \leq 2$. $\square$

Claim 14.3.3. *G is 2-connected.*

Proof. Suppose, for a contradiction, that $G \setminus u$ is disconnected for some $u \in V(G)$. Let C_1 be a component of $G \setminus u$. Let G_1 be the subgraph of G induced on $V(C_1) \cup \{v\}$ and let G_2 be a subgraph induced on $V(G) - V(C_1)$. G_1 and G_2 are connected, $|V(G_1)|, |V(G_2)| < |V(G)|$. Neither G_1 nor G_2 is a complete graph on $\Delta(G) + 1$ vertices because $\deg(u)$ in both G_1 and G_2 is smaller than $\Delta(G)$. So by the choice of G, we can colour G_1 and G_2 properly in at most $\Delta(G)$ colours. We can permute colours if necessary on G_1 and G_2, so that u has the same colour in both, and then we get a proper colouring of G. This is our contradiction. $\square$

Claim 14.3.4. *G is 3-connected*

Proof. Suppose, for a contradiction, that $G \setminus \{v_1, v_2\}$ is disconnected. Let C_1 be a component of $G \setminus \{v_1, v_2\}$, and let G_1 be obtained from a subgraph of G induced on $V(C_1) \cup \{v_1, v_2\}$ by adding an extra edge with ends u_1 and u_2, and let G_2 be obtained from a subgraph induced on $V(G) - V(C_1)$ in the same way.

As before, G_1 and G_2 are connected, $|V(G_1)|, |V(G_2)| < |V(G)|$. So by the induction hypothesis, we can colour G_1 and G_2 and permute their colours to match on u_1 and u_2, and obtain a colouring of G in $\Delta(G)$ colours, unless either G_1 or G_2 is a complete graph on $\Delta(G) + 1$ vertices.

Suppose WLOG that G_1 is complete on $\Delta(G) + 1$ vertices. Then, u_1 has a unique neighbour in $V(G_2) \setminus \{u_1, u_2\}$, say u_1'. Similarly, let u_2' be the unique neighbour of u_2. Then, $G \setminus \{u_1', u_2\}$ is also disconnected, and we ca consider graphs G_1' induced on $V(G_1) \cup \{u_1'\}$ and G_2' induced on $V(G_2) \setminus \{u_1\}$ (with an extra $u_1'u_2$ edge).

The same argument applies to G_1' and G_2' and neither of them is a complete graph on $\Delta(G) + 1$ vertices. This is because u_1' has degree z in G_1', and u_2 has degree z in G_2'. So we can colour G as before, completing the contradiction. $\square$

Now, for the main proof, choose $v \in V(G)$. If $\deg(v) < \Delta(G)$, then applying the greedy algorithm to the ordering from Claim 14.3.1 gives the required colouring. If every pair of neighbours of v is adjacent, then G is complete on $\Delta(G) + 1$ vertices, but G is not supposed to be. So let u_1, u_2 be a pair of non-adjacent neighbours of v. Apply Claim 14.3.1 to the graph $G \setminus \{u_1, u_2\}$ (this can be done by Claim 14.3.4), obtain an ordering $(v_1, v_2, \ldots, v)$. Now, consider the ordering $(u_1, u_2, v_2, \ldots, v)$. The greedy algorithm works for this ordering. $\square$

15 Edge Colouring

Definition. A *k-edge colouring* of a loopless graph G is a map $\varphi : E(G) \to S$ with $|S| = k$, so that $\varphi(e) \neq \varphi(f)$ if e and f share an end.

Definition. The minimum k such that G admits a k-edge colouring is called the *edge-chromatic number* of G and is denoted $\chi'(G)$.

Remark. One can show that $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.

Lemma 15.1. *Let G be a graph with $\Delta(G) \leq d$. Then, G is a subgraph of some d-regular graph H. Moreover, if G is loopless (or bipartite), then H can be chosen to be loopless (resp. bipartite).*

Proof. Let $V = V(G)$. Let V' contain a copy of every vertex of V. Let G' be a copy of G on V'. For all $v \in V$, if its copy is $v' \in V'$, join v to v' by $d - \deg(v)$ parallel edges in a new graph H. H is d-regular.

If G has a bipartition (A, B), and G' has a corresponding bipartition (A', B'), then $(A \cup B', A' \cup B)$ is a bipartition of H. □

Theorem 15.2 (König). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof. By Lemma 15.1, we may assume that G is k-regular for some k. We need to show that $\chi'(G) \leq k$. Equivalently, we want to show that the edge set of a k-regular bipartite graph ca be partitioned into k perfect matchings.

Recall, by Corollary 8.3, if G is bipartite k-regular, then G has a perfect matching. We induct on k. Once again, the base case is clear.

Let M be a perfect matching. Let $G' = G \setminus M$. Then, G' is $k - 1$ regular, and by induction it has $k - 1$ perfect matchings, and $\chi'(G') \leq k - 1$. By adding a extra colour for the edges of M, we obtain that $\chi'(G) \leq k$. □

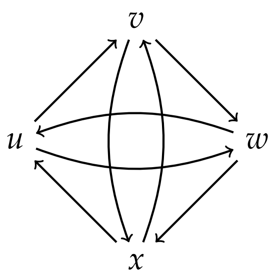
Remark. $\chi'(C_{2k+1}) = 3$, but $\Delta(C_{2k+1}) = 2$.

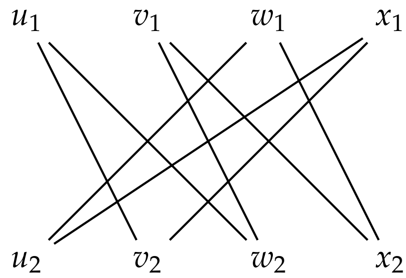
Definition. A *2-factor* in a graph G is a collection $F \subset E(G)$ such that every vertex is incident with exactly 2 edges.

Lemma 15.3. *Let G be a loopless $2k$-regular graph. Then, $E(G)$ can be partitioned into k 2-factors.*

Proof. By Lemma 6.2, we can partition $E(G)$ into a collection of cycles. Direct every cycle in some direction. Let G' be the resulting digraph. One can see that $\deg_+(v) = \deg_-(v) = k$ for every $v \in V(G')$, where $\deg_+(v) = |\delta^+(v)|$, and $\deg_-(v) = |\delta^-(v)|$, i.e. every vertex has k outgoing and k incoming edges.

Create a graph H from G' as follows: for every $v \in V(G)$, we create 2 vertices v_1, v_2 of H. If uv is an edge of G' directed from u to v, put an edge u_1v_2 into H.





$E(H)$ can be partitioned by Theorem 15.2 into k perfect matchings. Each perfect matching of H is a 2-factor in G. So we obtain the required partition. □

Theorem 15.4 (Shannon). *Let G be a loopless graph. Then $\chi'(G) \leq 3\lceil\Delta(G)/2\rceil$.*

Proof. Let $k = \lceil\Delta(G)/2\rceil$. Then, $\Delta(G) \leq 2k$, and by Lemma 15.1, G is a subgraph of a loopless $2k$-regular graph H. By Lemma 15.3, $E(G)$ can be partitioned into 2-factors $F_1, F_2, \ldots, F_k$. Let H_i be the graph with $E(H_i) = F_i$ and $V(F_i) = V(H)$. It is 2-regular. So connected components of H are cycles, and the edges of each of them can be 3-coloured. So $E(H)$ can be coloured in $3\lceil\Delta(G)/2\rceil$ colours. □

Theorem 15.5 (Vizing). *If G is simple, then $\chi'(G) \leq \Delta(G) + 1$.*

We omit this proof.

16 Series-Parallel Graphs

The process of *contracting an edge e* with ends u_1 and u_2 means deleting e and identifying u_1 and u_2.

Definition. A graph H is a *minor* of a graph G if it can be obtained from a subgraph of G by contracting edges sequentially.

It is important to note that being a minor is a transitive relation, i.e. if H' is a minor of H, and H is a minor of G, then H' is a minor of G.

A minor of G can equivalently be defined as a graph obtained from G by repeatedly deleting edges, deleting vertices, and contracting edges (in any order).

Part of this section is devoted to considering graphs with excluded minors. We begin with some easy examples.

Example.

- If G has no K_2 minor, then G only has loop edges.
- If G has no loop as a minor, then G is a forest.
- If G has no K_3 minor, then G does not have a cycle of length at least 3, i.e. G is obtained from a forest by adding loops and parallel edges.

Conjecture (Hadwiger)**.** *If G is loopless and does not have a K_n minor, then $\chi(G) \leq n-1$.*

This conjecture is known to hold for $n \leq 6$. It is also known to fail for infinite graphs. For $n = 2$, this says that if G has no edges, it is 1-colourable, which is trivial. For $n = 3$, this says that if G is a forest with parallel edges, it is 2-colorable. We know this to be true, since G would be bipartite.

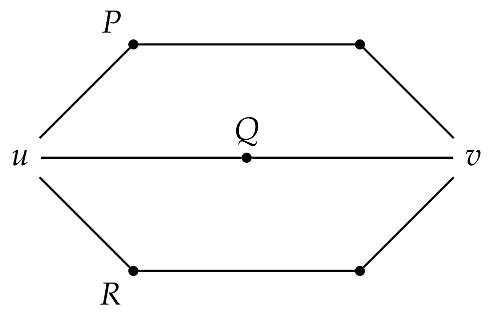
We will show that Hadwiger's conjecture holds for $n = 4$.

Definition. A graph G is a *subdivision* of H if edges of H are replaced in G by internally disjoint paths. In general, if G is a subdivision of H, then H is a minor of G.

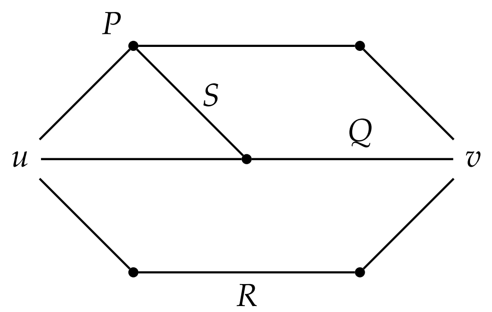
Lemma 16.1. *If G is 3-connected, then G has a K_4 minor.*

Proof. Since G is 3-connected, then $|V(G)| \geq 4$. If every 2 distinct vertices of G are adjacent, then G has a K_4 subgraph, so we assume that there are

are $u, v \in V(G)$ which are non-adjacent. By Corollary 9.3, there are paths P, Q, R with ends u and v, otherwise pairwise vertex disjoint.



The graph $G \setminus \{u, v\}$ is connected by 3-connectedness. Thus, there exists a path in $G \setminus \{u, v\}$ joining a vertex in the interior of one of the paths P, Q, R to a vertex in the interior of another path. Let S be the shortest such path. Let x and y be its ends. WLOG, $x \in V(P)$ and $y \in V(Q)$.



If z in the interior of S belongs to $V(P) \cup V(Q) \cup V(R)$, then S is not a shortest path, and a subpath of S with ends x and z or y and z contradicts the choice of S. So S is internally disjoint from P, Q, and R. Now, $P \cup Q \cup R \cup S$ is a subdivision of K_4, so G has a K_4 minor. □

Lemma 16.2. *Let G be a simple graph with no K_4 minor. Let X be a clique in G with $|X| \leq 2$. If $X \neq V(G)$, then there is a $v \in V(G) - X$ with $\deg_G(v) \leq 2$.*

Proof. We proceed by induction on $|V(G)|$. The base case is clear. In actuality, we can show that the theorem holds whenever $|V(G)| \leq 3$, and this is our base case.

So say $|V(G)| \geq 4$. By Lemma 16.1, G is not 3-connected, so there is a separation (A, B) of G with $|A \cap B| \leq 2$, $A \setminus B \neq \varnothing$, $B \setminus A \neq \varnothing$. Choose such a separation of minimal order. WLOG, $X \subset A$. Suppose $|A \cap B| = 2$. Then there exists a path between the vertices of $A \cap B$ in A, by minimality of $A \cap B$ (otherwise, a component of the subgraph induced on A containing one of the vertices of $A \cap B$ leads to a separation of smaller order).

Let G' be obtained from the subgraph of G induced on B by adding, if necessary, an extra edge joining vertices of $A \cap B$. Let $X' = A \cap B$. We would like to apply the induction hypothesis to G' and X'.

G' has no K_4 minor because G' is a minor of G, so there is a $v \in V(G') - X'$ with $\deg(v) \leq 2$, which satisfies the conditions for the lemma: $v \in B - A$, $\deg_{G'}(v) = \deg_G(v)$, $v \notin X \subset A$, $v \in V(G) - X$. □

Corollary 16.3. *Let G be a loopless graph with no K_4 minor. Then, $\chi(G) \leq 3$, i.e. Hadwiger's conjecture holds for $n = 4$.*

Proof. We proceed by induction on $|V(G)|$. The base case is clear.

We may assume that G is simple, since parallel edges do not affect colourability. By Lemma 16.2, G contains a vertex v with $\deg_G(v) \leq 2$. Consider $G' = G \setminus v$. Colouring G' by the induction hypothesis, we can then colour v in a colour distinct from its neighbours. □

Remark. Simple graphs with no K_4 minor are 2-generate, and therefore 3-colourable.

Definition. A graph G is *series-parallel* if it can be obtained from the null graph by repeatedly applying the following operations:

(1) adding a vertex of degree at most 1.

(2) adding a loop or a parallel edge.

(3) subdividing an edge.

Theorem 16.4. *Series-parallel graphs are exactly the graphs with no K_4 minor.*

Proof. Suppose G is series-parallel. One can show that a minor of a series parallel graph is series-parallel, and that K_4 is not series-parallel.

Conversely, suppose that G has no K_4 minor. We induct on $|V(G)| + |E(G)|$. The base case is clear.

If G contains a loop or parallel edge e, then G is constructed from $G \setminus e$ which is series-parallel by the induction hypothesis, by adding the edge e. So we may assume that G is simple. By Lemma 16.2, G contains a vertex v of degree at most 2. If this vertex v had degree at most 1, then G can be constructed from $G \setminus v$ by adding v. So we may assume that v has degree 2, in which case G can be constructed by subdividing an edge into v. □

17 Planar Graphs

A drawing of a graph G in the plane represents vertices as points in the plane, and edges as curves which do not intersect themselves or each other, and have ends at points corresponding to the ends of edges. Points in the plane not used in the drawing are divided into *regions*. Two points belong to the same region if they can be joined by a curve which avoids the drawing.

Some of the proofs in this section may sound slightly handwavy, but can likely be made formal with little effort by using the Jordan curve theorem.

Lemma 17.1. *Let G be a graph drawn in the plane, $e \in E(G)$. e is a cut edge if and only if the regions on either side of e are the same.*

Proof. Suppose the regions on both sides are the same. Then, there exists a curve avoiding the drawing from one side of e to the other. One can close this curve, separating the plane into 2 parts, with ends of e in different parts. So ends of e are in different components of $G \setminus e$. □

We prove the converse after establishing Euler's formula.

Definition. Let $\text{Reg}(G)$ denote the number of regions in the planar drawing of G.

Theorem 17.2 (Euler's Formula)**.** *Let G be a graph drawn in the plane. Then:*

$$|V(G)| - |E(G)| + Reg(G) = 1 + comp(G)$$

Proof. We induct on $|E(G)|$. The base case is trivial.

If G is a forest, then $|V(G)| - |E(G)| = \text{comp}(G)$ by Theorem 3.1. We need only check that $\text{Reg}(G) = 1$. Any closed walk in a forest uses every edge an even number of times. Thus, no edge can belong to different regions, since a boundary of a region is a collection of closed walks using every edge at most twice. Indeed, $\text{Reg}(G) = 1$.

If G is not a forest, the there is $e \in E(G)$ which is not a cut edge. Let $G' = G \setminus e$. Then, $\text{Reg}(G') = \text{Reg}(G) - 1$ by Lemma 17.1, so by the induction hypothesis:

$$\begin{aligned} |V(G)| - |E(G)| + \text{Reg}(G) &= |V(G')| - |E(G')| + \text{Reg}(G') \\ &= \text{comp}(G') + 1 = \text{comp}(G) + 1 \end{aligned}$$

□

Given this result, we complete the proof of Lemma 17.1.

Lemma 17.1. *Let G be a graph drawn in the plane, $e \in E(G)$. e is a cut edge if and only if the regions on either side of e are the same.*

Proof. If e is a cut edge, I claim that the regions on either side of e are the same. Let $G' = G \setminus e$. By Euler's formula:

$$\begin{aligned}\text{Reg}(G') &= 1 + \text{comp}(G') - |V(G')| + |E(G)| \\ &= 1 + \text{comp}(G) - |V(G)| + |E(G)| = \text{Reg}(G)\end{aligned}$$

So regions on both sides of e are the same. □

Definition. The *length of a region* is the number of edges of the graph in its boundary, where an edge is counted twice if the region is on both sides of the edge.

Lemma 17.3. *Let G be a connected simple graph, $|E(G)| \geq 2$. Then, every region has length at least 3.*

Proof. The only possible nontrivial closed walk of length at most 2 in a simple graph traces a single edge in both directions. Such a walk can form a boundary of a region only if the whole graph consists of 2 vertices and 1 edge. □

Lemma 17.4. *If G is simple planar, $|E(G)| \geq 2$, then $|E(G)| \leq 3|V(G)| - 6$, and if G contains no K_3 subgraphs, then $|E(G)| \leq 2|V(G)| - 4$.*

Proof. We may assume that G is connected, adding extra edges if necessary.

$$2|E(G)| = \sum_{\text{regions } r} \text{Length}(r) \geq 3\text{Reg}(G)$$

by Lemma 17.3, so $\text{Reg}(G) \leq \frac{2}{3}|E(G)|$, and:

$$\begin{aligned}&2 = |V(G)| - |E(G)| + \text{Reg}(G) \leq |V(G)| - \frac{|E(G)|}{3} \\ &6 \leq 3|V(G)| - |E(G)|\end{aligned}$$

If G contains no K_3 subgraph, then G has no regions of length 3, so $\frac{|E(G)|}{2} \geq \text{Reg}(G)$, and the rest is clear. □

Definition. Let $K_{m,n}$, a *complete bipartite graph*, denote the bipartite simple graph on $m + n$ vertices with bipartition (A, B), $|A| = m$, $|B| = n$, and the maximum number of edges, i.e. every vertex of A is joined to every vertex of B.

Corollary 17.5. *Graphs K_5 and $K_{3,3}$ are not planar.*

Proof. A simple application of Lemma 17.4. □

Corollary 17.6. *Let G be simple, planar, and $|E(G)| \geq 2$. Let n_i denote the number of vertices of G of degree i. Then:*

$$\sum_i (6 - i)n_i \geq 12$$

and

$$\sum_{v \in V(G)} (6 - \deg(v)) \geq 12$$

Proof. From Lemma 17.4, we have:

$$\begin{aligned} 12 &\leq 6|V(G)| - 2|E(G)| \\ 12 &\leq 6\sum_i n_i - \sum_i in_i = \sum_i (6 - i)n_i \end{aligned}$$

The second inequality follows immediately. □

18 Kuratowki's Theorem

Lemma 18.1. *Let G be a 2-connected loopless planar graph. Then, every region in the drawing is bounded by a cycle of G.*

Proof. Every region is bounded by a closed walk. Suppose, for a contradiction, that some vertex v is repeated in the boundary walk of some region. Then there exists a closed curve in the plane intersecting the drawing at v and separating the drawing into two nonempty parts. This corresponds to a separation of G, (A, B), with $A \cap B = \{v\}$ and $A, B \neq V(G)$, which contradicts 2-connectivity. □

We now prove a lemma which is seemingly unrelated to planarity, but which will become useful in proving Kuratowski's theorem later.

Lemma 18.2. *Let C be a cycle. Let $X, Y \subset V(C)$. Then, one of the following holds:*

(1) There are $u, v \in V(C)$, such that if P, Q are paths from u to v forming C, then $X \subset V(P)$ and $Y \subset V(Q)$.

(2) There are distinct $x_1, x_2 \in X$, $y_1, y_2 \in Y$, such that x_1, y_1, x_2, y_2 appear on C in this order.

(3) $X = Y$, and $|X| = 3$.

Proof. We may assume that $|X|, |Y| \geq 2$. Otherwise, (1) easily holds. We may also assume that $X \neq Y$. WLOG, there is a vertex $x_1 \in X - Y$. let $y_1, y_2 \in Y$ be the first vertices in Y we encounter from x_1 following C in either direction. Since $|Y| \geq 2$, $y_1 \neq y_2$. Let P and Q be the two paths into which y_1 and y_2 separate C, $x_1 \in V(P)$. If there is a vertex of X in the interior of Q, then (2) holds. Otherwise, (1) holds. □

Theorem 18.3 (Kuratowski-Wagner). *G is planar if and only if G does not contain either K_5 or $K_{3,3}$ as a minor.*

Proof. By Corollary 17.5, K_5 and $K_{3,3}$ are nonplanar, and a minor of a planar graph is planar.

Conversely, we proceed by induction on $|V(G)| + |E(G)|$. The base case is clear. For induction, we assume that G is simple. Similarly, we see that G is connected.

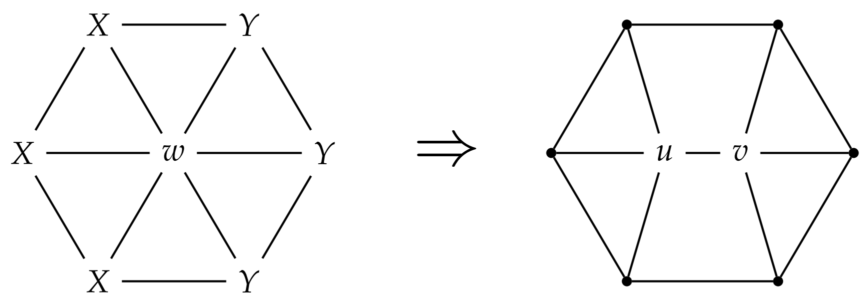
Suppose that G is not 2-connected, and (A, B) is a separation of G with $A \cap B = \{v\}$, $A, B \neq V(G)$. Let G_1, G_2 be subgraphs of G induced on A and B respectively. Apply the induction hypothesis to G_1 and G_2. If either of them has K_5 or $K_{3,3}$ as a minor, then so does G. So we assume that neither contains K_5 or $K_{3,3}$ as a minor, so G_1 and G_2 are planar. Then, we can "glue" these drawings at v to obtain a drawing of G.

So it suffices to consider when G is 2-connected. If G is not 3-connected, the argument is similar: if (A, B) is a separation with $A \cap B = \{u, v\}$, $A, B \neq V(G)$. Add a edge e between u and v to G, and let G_1 and G_2 be the subgraphs of the resulting graph induced by A and B respectively. As there is a path from u to v in G inside A and inside B, G_1 and G_2 are minors of G. As before, we assume that G_1 and G_2 can be drawn in the plane, and we can obtain a drawing of G by gluing them alone e.

So we may assume that G is 3-connected. Finally, suppose that for some edge e with ends u and v, $G \setminus \{u, v\}$ is not 2-connected (G is nearly 4-connected). Then, let (A, B) be a separation of G with $A \cap B = \{u, v, w\}$, and $A, B \neq V(G)$. Let G_1 and G_2 be obtained from subgraphs of $G \setminus e$ induced on A and B respectively by adding an extra vertex joined to u, v and w. G_1 and G_2 are minors of G, since every component of $G \setminus A$ has edges going to all 3 of u, v and w, so it can be contracted to an extra vertex to form G. Apply the induction hypothesis to G_1 and G_2.

Let e with ends u, v be any edge of G. Let G' be obtained by contracting e, and let w be the vertex obtained by identifying u and v. If G' is not planar, we are done. So G' is planar, and consider its planar drawing. Consider the drawing of $G' \setminus w = G' \setminus \{u, v\}$ inside our drawing. Since $G' \setminus w$ is 2-connected, the region containing w is bounded by a cycle of G, say C. Let $X \subset V(C)$ be the set of neighbours of u, and let Y be the set of neighbours of v. By Lemma 18.2, one of several outcomes can occur:

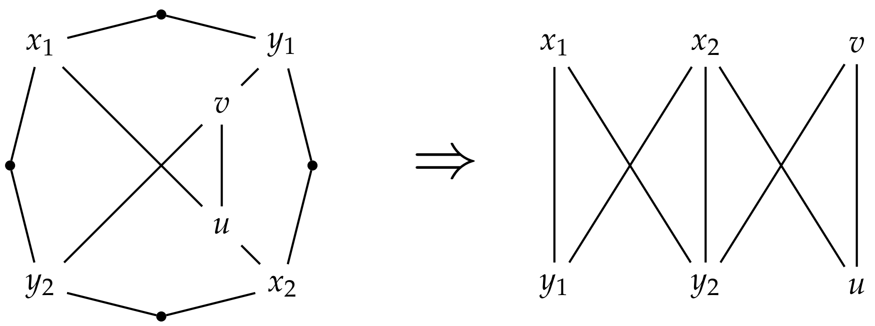
(1) There are $s, t \in V(C)$ such that if P and Q are paths with ends s and t forming C, then $X \subset P$ and $Y \subset Q$:



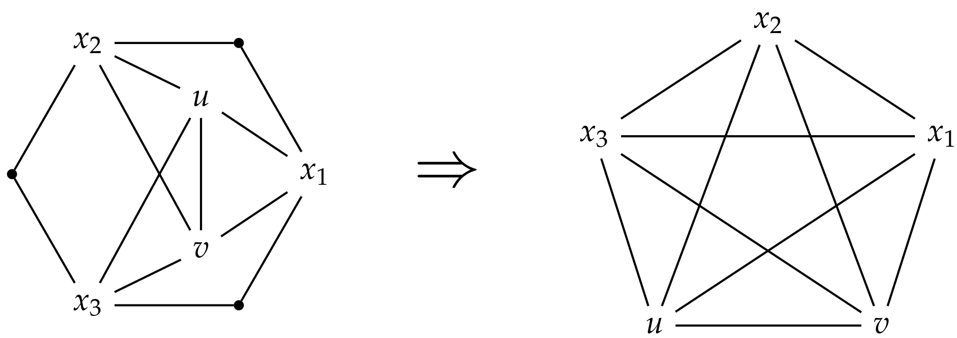
and G is planar.

(2) There are $x_1, x_2 \in X$, $y_1, y_2 \in Y$, appearing on C in the order x_1, y_2, x_2, y_2.

This is a subdivision of $K_{3,3}$ with vertex sets $\{u, y_1, y_2\}$ and $\{v, x_1, x_2\}$:



(3) $X = Y$, $|X| = 3$. In this case, let $X = \{x_1, x_2, x_3\}$. Now, G contains a subdivision of K_5 with vertices u, v, x_1, x_2, x_3, so G is non-planar and contains a K_5 minor:



So the proof is complete. □

Theorem 18.4 (Kuratowski)**.** *G if planar if and only if G does not contain a subdivision of either K_5 or $K_{3,3}$ as a subgraph.*

Proof. Containing a graph as a subdivision implies containment as a minor.

Conversely, we induct on $|V(G)|$. There exists a sequence:

$$G \to G_1 \to G_2 \to \ldots \to G_k$$

of graphs, starting with G and ending with K_5 or $K_{3,3}$, where each subsequent graph is obtained by edge or vertex deletion, or edge contraction. By the induction hypothesis, G_1 contains a subdivision of K_5 or $K_{3,3}$. If the last operation is not contraction, then G_1 is a subgraph of G and so is a subdivision.

Otherwise, let w be a vertex obtained by contracting an edge. If $\deg(w)$ in the subdivision of K_5 or $K_{3,3}$ in G is 3 or less, then replacing w by u and v, and adding an edge uv to the subdivision, yields a subdivision of G.

The only case when we do not retrieve the original subdivision is when $\deg(w) = 4$, with 2 edges corresponding to edges incident to u, and 2 incident to v. In this case, we obtain a subdivision of $K_{3,3}$. □

One can also consider drawing graphs on surfaces other than $\mathbb{R}^2$. For instance, consider the real projective plane $\mathbb{P}^2$, which is formed from a circle by identifying antipodal points. This surface cannot be embedded in $\mathbb{R}^3$, but we may still consider non-self intersecting drawings of graphs embedded in the projective plane.

Theorem 18.5 (Archdeacon)**.** *There exist 35 graphs such that every non-projective planar graph contains one as a minor.*

Remark. 103 graphs need to be excluded as subdivisions.

Remark. There are over 16000 excluded minors if one considers non-self intersecting graphs embedded in the torus.

19 Colouring Planar Graphs

Theorem 19.1 (Heawood)**.** *Let G be planar and loopless. Then $\chi(G) \leq 5$.*

Proof. We induct on $|V(G)|$. If G has a vertex of degree at most 4, we are done by induction. Otherwise, by Corollary 17.6, G has a vertex of degree 5. Let u_1, u_2, u_3, u_4, u_5 be its neighbours. WLOG, u_1 and u_3 are non-adjacent, otherwise G would contain a K_6 subgraph. Let G' be obtained from G by deleting v and identifying u_1 and u_3.

G' is a minor of G, so G' is planar, and $\chi(G') \leq 5$ by the induction hypothesis. Let $G \setminus v$ inherit the colouring of G'. Then, u_1 has the same colour as u_3, so neighbours of v have at most 4 different colours among them, so $\chi(G) \leq 5$. □

Definition. A *planar triangulation* is a planar graph in which every region, including the infinite one, is a triangle.

Theorem 19.2 (Appel-Haken, The Four Colour Theorem)**.** *If G is planar and loopless, then $\chi(G) \leq 4$.*

Proof. The proof is computerized, but we give an outline which describes, roughly, how it works: one considers a finite set $\mathcal{X}$ with the following properties:

(1) Every sufficiently well-connected planar triangulation contains one of the graphs in $\mathcal{X}$.

(2) If G is planar and contains a graph $H \in \mathcal{X}$, then H can be replaced by a smaller graph in G so that if the resulting graph G' is 4-colourable, then so is G.

□

Lemma 19.3. *Let G be a simple planar triangulation. Then, G contains one of the following:*

(1) A vertex of degree at most 4.

(2) 2 neighbours of degree 5.

(3) A region with vertices having degrees 5, 6 and 6.

Proof. We proceed by black magic. Let us assign to every vertex of G, a "charge" $6 - \deg(v)$. Let us also assume that the minimum degree of any vertex in G is at least 5. By Corollary 17.6, the sum of charges is at least

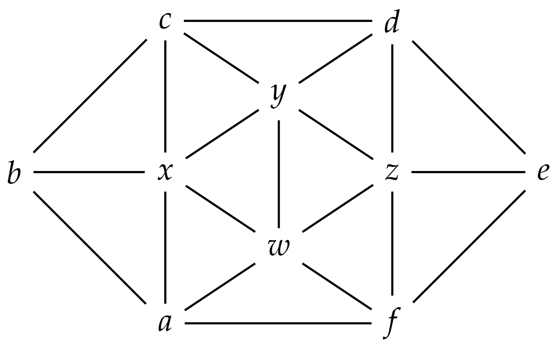
12. Let every vertex of degree 5 distribute its charge uniformly among its neighbours of degree at least 7. This is the "discharging rule." After this redistribution, some vertex v has positive charge. At this point, we note that the lemma holds unless every vertex of degree 5 has at least 3 neighbours of degree at least 7, so suppose this is the case.

Every vertex receives charge $\leq \frac{1}{3}$ from each of its degree 5 neighbours. At most $\lfloor \deg(v)/2 \rfloor$ neighbours have degree 5, since neighbours form a cycle and no two are adjacent. Its charge is at most:

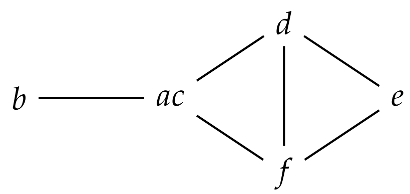
$$(6 - \deg(v)) + \frac{1}{3}\lfloor \deg(v)/2 \rfloor \leq 0$$

if $\deg(v) \geq 7$, which is a contradiction. □

Lemma 19.4. *Let G be a planar triangulation containing the following subgraph:*

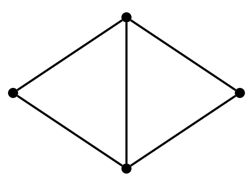


Let G' be obtained from G by deleting w, x, y, z and identifying a and c, and including an edge between f and d.

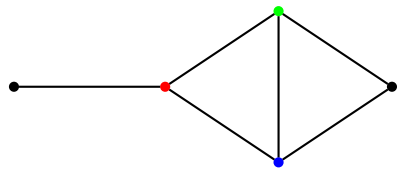


Then, if G' is 4-colourable, so is G.

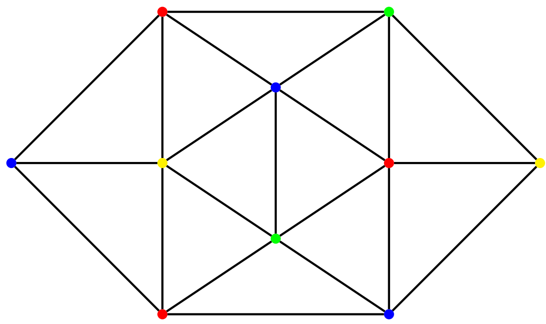
Proof. One should note that such a subgraph is implied from the containment of the following, simpler subgraph:



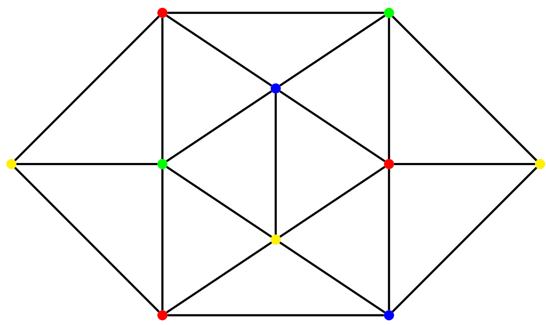
We continue the proof by consider colourings in red, blue, green, and yellow. We can assume, WLOG, that G' has the following partial colouring:



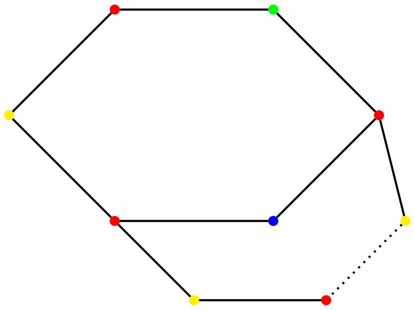
Suppose b is not coloured yellow in G''s 4-colouring. We may assume that it is blue, by symmetry. The colouring extends to G as follows:



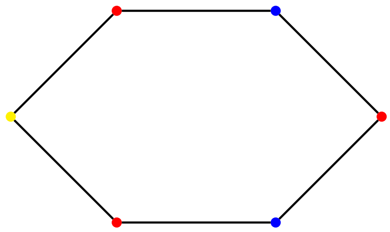
Otherwise, suppose b is coloured yellow in G''s 4-colouring. Suppose e is coloured yellow as well. The colouring extends to G as follows:



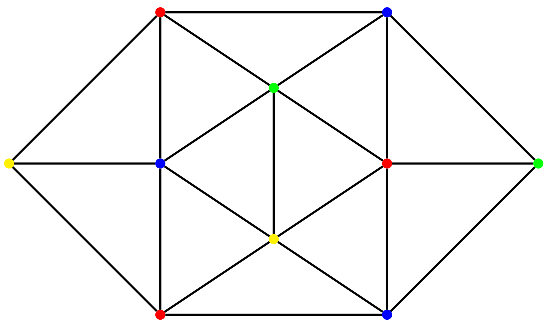
Finall, suppose b is coloured yellow and e is coloured red. We cannot so easily extend this colouring to a 4-colouring of G. Instead, consider the subgraph of G induced by colours red and yellow. If e and $\{a, b, c\}$ belong to different components of this subgraph, then swap red and yellow in the component of e. Then, G is 4-colourable, from previous case analysis, so e and $\{a, b, c\}$ belong to the same component, and there exists a path from e to $\{a, b, c\}$.



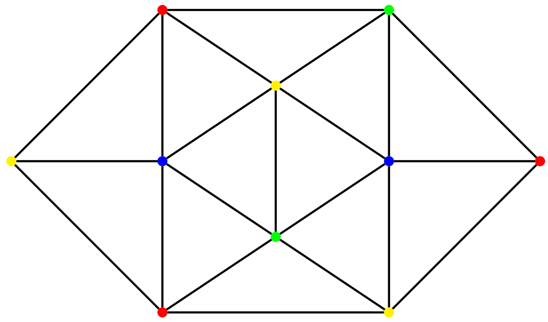
Therefore, f and d belong to different components of the graph induced by blue and green colours. Swapping blue and green on the component containing f yields:



This colouring does not yet extend to vertices w, x, y and z. Consider the component of red and green containing e in the resulting graph. If it contains neither a nor c, then swap red and green within this component. Then, G is 4-colourable:



If, finally, the component contains, say, a and e, then f belongs to different components of yellow and green than either b or d, so swap yellow and green on the component of f to obtain a 4-colouring of G:



So G is 4-colourable. $\square$

Definition. Let G be a connected planar graph. Let G^* be another planar graph. We say that G^* is a *planar dual* of G if:

(1) Every region of G contains exactly one vertex of G^*.

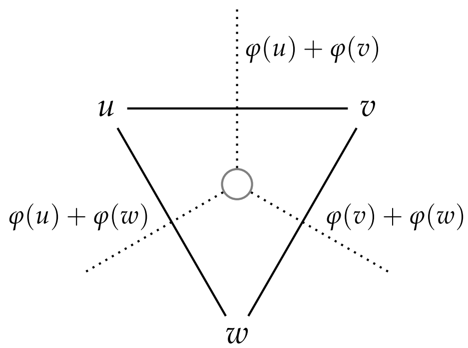
(2) Every edge of G is crossed by exactly one edge of G^*, and the drawings of G and G^* are otherwise disjoint.

(3) $|E(G)| = |E(G^*)|$.

Theorem 19.5 (Tait)**.** *Let G be a planar triangulation. Let G^* be a planar dual of G. Then, $\chi(G) \leq 4$ if and only if $\chi'(G^*) = 3$.*

Proof. What follows is a very interesting proof.

We first show that 4-colourability implies 3-edge colourability of the planar dual. Consider colouring vertices of G by elements of the Klein 4-group, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\varphi : V(G) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be a proper 4-colouring. Define $\psi : E(G^*) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \setminus \{(0,0)\}$ as follows: if $e \in E(G^*)$ crosses an edge of G joining u and v, let $\psi(e) = \varphi(u) + \varphi(v)$.



This gives a proper colouring, since $\varphi(u) \neq \varphi(v) \neq \varphi(w)$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Conversely, suppose that $\chi'(G^*) = 3$. Let $\psi : E(G^*) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \setminus \{(0,0)\}$ be a proper edge colouring of G^*. Let $F_{a,b} = \psi^{-1}(a,b)$ be the colour classes in this edge colouring. Consider the subgraph H_1 of G^* with edges $E(H_1) = F_{1,0} \cup F_{1,1}$. Similarly, let H_2 be a subgraph of G^* with edges $E(H_2) = F_{0,1} \cup F_{1,1}$. H_i is 2-regular, so it is a collection of cycles. The plane can be 2-coloured so that every cycle in H_i separates regions of different colours.

Let $\varphi : V(G) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be such that $\pi_i \varphi(v)$ corresponds to the colour of a region of H_i that v is in. I claim that φ is a proper colouring. Indeed, let $u, v \in V(G)$ be adjacent, and let e be an edge between them. Let e^* be the corresponding edge of G^*. WLOG, $e^* \in H_1$, so $\varphi(u)$ and $\varphi(v)$ different in the first coordinate, and $\varphi(u) \neq \varphi(v)$. So φ is a proper colouring, and G is 4-colourable. □

20 Perfect Graphs

We assume, unless stated otherwise, that all graphs in this section are simple.

Definition. A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Intuitively, one might motivate this definition as follows: the most "difficult" graphs to colour are the complete graphs, so if colouring G can be done in a "nice" way by colouring its cliques, then it is perfect.

Example. Edgeless graphs, complete graphs, and bipartite graphs are perfect.

Definition. If G is simple, then $\overline{G}$ is called a complement of G if $V(\overline{G}) = V(G)$, and for $u \neq v \in V(\overline{G})$, $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Lemma 20.1. *Complements of bipartite graphs are perfect.*

Proof. Let G be a bipartite graph. Note that $\omega(\overline{G}) = \alpha(G)$. Also, any colour class in a proper colouring of $\overline{G}$ has size at most 2. Let $\mathcal{M}$ be the set of colour classes of size 2:

$$\chi(\overline{G}) = |V(G)| - |\mathcal{M}| = |V(G)| - \nu(G)$$

since colour classes of size 2 form a matching in G. Then:

$$\omega(\overline{G}) = \chi(\overline{G}) \iff \alpha(G) = |V(G)| - \nu(G)$$

and we know this to hold since G is bipartite, from Lemma 11.1 and König's theorem.

Since induced subgraphs of $\overline{G}$ are again, complements of bipartite graphs, then $\chi(H) = \omega(H)$ for any induced subgraph of $\overline{G}$. □

Remark. If G is bipartite, $L(G)$ is its line graph, then $\chi(L(G)) = \chi'(G)$, $\omega(L(G)) = \Delta(G)$. We already know that $\chi'(G) = \Delta(G)$ for bipartite graphs by Theorem 15.2, so line graphs of bipartite graphs are perfect.

Lemma 20.2. *Complements of line graphs of bipartite graphs are perfect.*

Proof. Let G be bipartite. Then, $\omega(\overline{L(G)}) = \nu(G)$. An independet set in $\overline{L(G)}$ is a clique in $L(G)$, which is a set of edges incident to a vertex of G. Colouring $\overline{L(G)}$ means partitioning $\overline{L(G)}$ into independent sets, i.e. finding a collection of vertices of G such that every edge is incident to one of them. So $\chi(\overline{L(G)}) = \tau(G)$. Since G is bipartite, then $\chi(\overline{L(G)}) = \omega(\overline{L(G)})$. □

Remark. Perfect graphs do not have odd cycles of length at least 5 as induced subgraphs.

Definition. A graph is *chordal* if it has no cycles of length at least 4 as induced subgraphs.

Example. Trees, complete graphs, and complete graphs "glued" in a tree-like fashion are all chordal.

Definition. Let $G = H_1 \cup H_2$ and let $S = H_1 \cap H_2$. Then, we say that G is obtained from H_1 and H_2 by *gluing* along S.

Theorem 20.3. *A graph is chordal if and only if it can be obtained by repeatedly gluing along complete subgraphs starting with complete graphs.*

Proof. Let $G = H_1 \cup H_2$, with H_1 and H_2 chordal, and $S = H_1 \cap H_2$ complete. We will show that G is chordal. Suppose not, i.e. G contains an induced cycle C of length at least 4. Then, it is easy to see that $C \subset H_1$, or $C \subset H_2$, but neither can happen, since H_1 and H_2 are chordal.

Conversely, it suffices to show that any chordal graph G is either complete, or can be obtained from 2 smaller graphs by gluing along a complete graph. Suppose G is not complete. Let $u, v \in V(G)$ be nonadjacent. Let $P_1, P_2, \ldots, P_k$ be a collection of internally disjoint paths in G with ends u and v, where k is made to be maximal.

By Theorem 9.2, Menger's theorem, there exists a separation (A, B) of G such that $|A \cap B| = k$, $u \in A - B$, $v \in B - A$. If we show that vertices of $A \cap B$ are pairwise adjacent, then we would deduce that G is obtained by gluing subgraphs induced by A and B along a complete graph, as desired.

Suppose not. Say $x_1, x_2 \in A \cap B$ are not adjacent. WLOG, $x_1 \in V(P_1)$, $x_2 \in V(P_2)$. Consider the subgraph induced by $(V(P_1) \cup V(P_2)) \cap A$. In this subgraph, there exists an induced path Q with ends x_1 and x_2. Similarly, let R be an induced path between x_1 and x_2 in the subgraph induced by $(V(P_1) \cup V(P_2)) \cap B$. As x_1 and x_2 separate $V(Q)$ from $V(R)$, we have that $V(Q) \cup V(R)$ induces a cycle of length at least 4 in G, contradicting the fact that G is chordal. So $A \cap B$ induces a complete subgraph. □

Remark. Induced subgraphs of chordal graphs are chordal.

Corollary 20.4. *Chordal graphs are perfect.*

Proof. We induct on $|V(G)|$. Let G be a chordal graph. It suffices to show that $\omega(G) = \chi(G)$, because being chordal is preserved by induced subgraphs.

Either G is complete and perfect, or by Theorem 20.3, $G = H_1 \cup H_2$ with $S = H_1 \cap H_2$ a clique, and H_1, H_2 are perfect. If H_1 is coloured in $\chi(H_1)$ colours, then we can assume that $V(S)$ is coloured in colours $\{1, 2, \ldots, |V(S)|\}$ in an appropriate order. Combine the colouring of H_1 with a colouring of H_2 in $\chi(H_2)$ colours. Then:

$$\chi(G) = \max\{\chi(H_1), \chi(H_2)\} = \max\{\omega(H_1), \omega(H_2)\} \leq \omega(G)$$

Since $\chi(G) \geq \omega(G)$, for any graph G, then $\omega(G) = \chi(G)$. □

Definition. Let G be a graph, $x \in V(G)$, and let G' be obtained from G by adding a extra vertex x' and joining it by edges to x and all the neighbours of x. Then, we say that G' is obtained from G by *expanding x to an edge xx'*.

Lemma 20.5. *If G is perfect and G' is obtained from G by expanding $x \in V(G)$ to an edge xx', then G' is perfect.*

Proof. We induct on $|V(G)|$. It is enough to show that $\chi(G') \leq \omega(G')$. We know that $\chi(G) = \omega(G)$. If $\omega(G') = \omega(G) + 1$, then $\chi(G') \leq \chi(G) + 1 \leq \omega(G')$, and we are done. So say $\omega(G') = \omega(G)$, and we want to show that $\chi(G') \leq \omega(G)$.

Every clique in G containing v can be extended to a larger clique in G'. Therefore, v is in no maximum clique in G. G is perfect, so it can be coloured in $\omega(G)$ colours. Let $S \subset V(G)$ be the colour class of v. Then, ever clique of size $\omega(G)$ in G contains a element of $S - \{v\}$. Consider $G - (S - \{v\})$. We have:

$$\omega(G - (S - \{v\})) \leq \omega(G) - 1$$

By perfection of G, then:

$$\chi(G - (S - \{v\})) \leq \omega(G) - 1$$

Now, consider the remaining vertices in G', $(S - \{v\}) \cup \{v'\}$. Note that v' has no neighbours in S, since v has none. So the remaining vertices form an independent set and can all be coloured the same colour. So $\chi(G') \leq \omega(G)$, as desired. □

Remark. By Lemma 20.5, a graph obtained from a perfect graph by replacing each vertex by a complete subgraph is perfect.

Theorem 20.6 (Lovász, Weak Perfect Graph Theorem)**.** *G is perfect if and only if $\overline{G}$ is perfect.*

Proof. We proceed by induction on $|V(G)|$. It suffices to show that $\chi(\overline{G}) = \omega(\overline{G}) = \alpha(G)$. Suppose that there exists a clique $K \subset V(G)$ which intersects every $A \subset V(G)$, such that A is an independent set with $|A| = \alpha(G)$. Then, $\chi(\overline{G}) \leq \chi(\overline{G} \setminus K) + 1$ by colouring K in a single colour. By the induction hypothesis:

$$\chi(\overline{G}) \leq \omega(\overline{G} \setminus K) + 1 = \alpha(\overline{G} \setminus K) + 1 \leq \alpha(G)$$

This suffices.

To complete the proof, we refer to more black magic. Let $\mathcal{K}$ be the set of all cliques in G. Let $\mathcal{A}$ be the set of all maximum independent sets. By what we have shown above, we may assume that for all $K \in \mathcal{K}$, there exists an $A_K \in \mathcal{A}$ such that $K \cap A_K = \varnothing$. For $v \in V(G)$, let:

$$k(v) = |\{K : v \in A_K, K \in \mathcal{K}\}|$$

Let G' be obtained from G by replacing each vertex $v \in V(G)$ by a clique on $k(v)$ vertices. By Lemma 20.5, G' is perfect.

There exists a clique $X \in \mathcal{K}$ such that:

$$\begin{aligned}\omega(G') &= \sum_{v \in X} k(v) = \sum_{v \in X} |\{K : v \in A_K, K \in \mathcal{K}\}| \\ &= \sum_{K \in \mathcal{K}} |\{v \in X : v \in A_K\}| = \sum_{K \in \mathcal{K}} |X \cap A_K| \leq |\mathcal{K}| - 1\end{aligned}$$

since $|X \cap A_K| \leq 1$, and $|X \cap A_X| = 0$. In addition, $\alpha(G') \leq \alpha(G)$:

$$\begin{aligned}|V(G')| &= \sum_{v \in V(G)} k(v) = \sum_{v \in V(G)} |\{K : v \in A_K, K \in \mathcal{K}\}| \\ &= \sum_{K \in \mathcal{K}} |A_K| = \alpha(G)|\mathcal{K}|\end{aligned}$$

If G' is perfect, it can be coloured in $\omega(G') \leq |\mathcal{K}| - 1$ colours, but each class has size at most $\alpha(G') \leq \alpha(G)$, so altogether, we can colour at most $\alpha(G)(|\mathcal{K}| - 1) < \alpha(G)|\mathcal{K}| = |V(G')|$ vertices, which is a contradiction since any colouring colours all vertices. ☐

Remark. Which graphs are "minimally imperfect?" We know that odd cycles of length at least 5 are minimally imperfect. Their complements are also minimally imperfect, by Theorem 20.6.

Theorem 20.7 (Chudnovsky, Robertson, Seymour, Thomas, Strong Perfect Graph Theorem)**.** *A graph G is perfect if and only if it does not contain C_{2k+1} or $\overline{C_{2k+1}}$ for $k \geq 2$ as an induced subgraph.*

21 Stable Matchings and List Colouring

Definition. A *system of preferences* is an assignment, for every $v \in V(G)$, of a linear order on a set of its neighbours. We write $a >_c b$ to mean that "c prefers a to b."

Definition. A *stable matching* in a graph with a system of preferences is a matching M satisfying the following: if a and b are vertices joined by an edge in $E(G) - M$, then it is impossible that $b >_a v$, where v is the vertex matched to a in M (if any), and $a >_b u$, where u is the vertex matched to b in M (if any).

Theorem 21.1 (Gale-Shapley)**.** *Any bipartite simple graph with a system of preferences has a stable matching.*

Proof. Let G be a bipartite graph with bipartition (M, W). For clarity (and comic relief), vertices in M are men and vertices in W are women. We produce an algorithm which will output a stable matching.

input: Bipartite graph G with bipartition (M, W).

algorithm:

At each step, each man proposes to the woman he likes most, and which has not rejected him yet. Each woman rejects all men who propose to her, except for the one among them she likes the most. To this one, she says maybe. If no woman receives > 1 proposal, every woman marries the man proposing to her, thus creating a matching.

output: A stable matching.

We claim that this algorithm terminates in a stable matching. Indeed, the algorithm terminates, since at each non-terminal step in the algorithm, some man gets rejected, but the total number of rejections cannot exceed $|E(G)|$, so the algorithm terminates in at most $|E(G)|$ steps.

Let F be the resulting matching. Suppose that $m \in M$, $w \in W$ are joined by an edge in $E(G) - F$. If m proposed to w and got rejected, then at that point, w prefers another man proposing to her, so she eventually is matched with a man she prefers to m. If m did not propose to w, then m was always proposing to somebody he prefers to w, and eventually was matched to somebody he prefers to w. So indeed, F is a stable matching. $\square$

Remark. Using the classical approach to marriage proposals (i.e. men proposing), this algorithm yields the best results for men and the worst results for women.

Definition. Let G be simple. Let $\mathcal{S} = \{S(v) : S(v) \subset \mathbb{Z}, v \in V(G)\}$ be an assignment of lists $S(v)$ of colours to each vertex of the graph. We say that $c : V(G) \to \mathbb{Z}$ is an $\mathcal{S}$*-colouring* if $c(v) \in S(v)$ and it is a proper colouring.

We say that G is *k-list-colourable* if G is $\mathcal{S}$-colourable for any collection of lists $\{S(v) : v \in V(G)\}$ such that $|S(v)| \geq k$ for every $v \in V(G)$.

Definition. The *list-chromatic number* $\chi_L(G)$ is the minimum k such that G is k-list-colourable.

Remark. For any graph, $\chi_L(G) \geq \chi(G)$. Also, $\chi_L(K_{3,3}) > \chi(K_{3,3}) = 2$.

Definition. Let G be a graph, $\mathcal{S} = \{S(e) : e \in E(G), S(e) \subset \mathbb{Z}\}$. We say that $c : E(G) \to \mathbb{Z}$ is an $\mathcal{S}$*-list-colouring* if $c(e) \in S(e)$ and $c(e) \neq c(f)$ if e and f are distinct edges sharing an end.

We say that G is *k-edge-list-colourable* if it is $\mathcal{S}$-list-colourable for every collection of sets $\mathcal{S}$ such that $|S(e)| \geq k$ for every $e \in E(G)$.

Definition. The *edge-list-chromatic number* of $\chi'_L(G)$ is the smallest k such that G is k-edge-list-colourable.

Conjecture 21.2. *For every simple graph G, $\chi'(G) = \chi'_L(G)$.*

Lemma 21.3. *Let G be a bipartite simple graph with bipartition (A, B), equipped with a system of preferences. For every edge $e \in E(G)$, with ends $u \in A$, $v \in B$, let $\varphi(e)$ be the number of neighbours of u which u rfers to v + the number of neighbours of v which v prefers to u. Let $\mathcal{S} = \{S(e) : e \in E(G), S(e) \subset \mathbb{Z}\}$ such that $|S(e)| \geq \varphi(e) + 1$ for every edge $e \in E(G)$. Then G is $\mathcal{S}$-edge-list-colourable.*

Proof. We induct on $|E(G)|$. The base case is trivial.

Let $c \in \mathbb{Z}$ be a colour appearing in $S(e)$ for some $e \in E(G)$. let H be a subgraph of G with the same vertex set, and $E(H)$ consisting of edges which include c in their lists. Let M be a stable matching in H, which exists by Theorem 21.1. Colour edges of M in the colour c. We would like to colour the rest by the induction hypothesis.

Let $G' = S \setminus M$, let $S'(e) = S(e) - \{c\}$ for $e \in E(G')$. It suffices to check that $|S'(e)| \geq \varphi'(e) + 1$ for $e \in E(G')$, where φ' is calculated as φ, but on G' instead of G.

First, suppose that $c \notin S(e)$. Then:

$$|S'(e)| = |S(e)| \geq \varphi(e) + 1 \geq \varphi'(e) + 1$$

Otherwise, $c \in S(e)$, and $|S'(e)| = |S(e)| - 1$. So we want to show that $\varphi'(e) \leq \varphi(e) - 1$. This is equivalent to saying that M contained an edge which is counted by $\varphi(e)$, i.e. joins one of the ends, say u, of e to the vertex u prefers. This is true by definition of a stable matching. □

Theorem 21.4 (Galvin)**.** *Conjecture 21.2 is true for simple bipartite graphs.*

Proof. We know, by Theorem 15.2, that $\chi'(G) = \Delta(G)$ for bipartite G. Let $c : E(G) \to \{1, 2, \ldots, k\}$ be an edge-colouring of G, with $k = \Delta(G)$. Let (A, B) be the bipartition of G. For a vertex $u \in A$, let u prefer neighbours to which it is joined by edges of smaller colour. Let vertices in B prefer neighbours t which they are joined by edges of larger colours.

Let $\mathcal{S} = \{S(e) : e \in E(G), S(e) \subset \mathbb{Z}\}$ such that $|S(e)| \geq k$ for every e. By Lemma 21.3, it suffices to show that $\varphi(e) \leq k - 1$ for every edge e, where φ is in the statement of Lemma 21.3.

Suppose $c(e) = i$. Let $u \in A, v \in B$ be its ends. Then, u prefers $\leq i - 1$ neighbours to v, and v prefers $\leq k - i$ neighbours to u, so:

$$\varphi(e) \leq (i - 1) + (k - i) = k - 1$$

as desired. □

List of Theorems