Name ______________________

MATH 350: Graph Theory and Combinatorics. Fall 2017.
Midterm Exam

Thursday, October 19th, 2017, 8:40-9:55

The questions have to be answered in the booklets provided.

You can choose which two questions to answer. Indicate your choice on the front page. Only the two chosen questions will be graded. Each question is worth 50 points.

Write your answers clearly. Fully justify all your answers.

You can consult your notes and textbooks. Use of calculators, computers, cell-phones, etc. is not permitted.

Problem	Your choice	Your score
1		
2		
3		
Total		

1. Let $\mathcal{G}$ be the set of all simple non-isomorphic graphs G satsfying:

- G has ten vertices in total,
- G has exactly four vertices of degree 3,
- G has exactly four vertices of degree 4, and
- G has exactly two vertices of degree 5.

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a) Construct a graph $G \in \mathcal{G}$ that is bipartite. *(10 points)*

b) Is there a bipartite graph $G \in \mathcal{G}$ with a bipartition (A, B) satisfying $|A| \neq |B|$?
(If yes, construct a one. If no, prove it!) *(10 points)*

c) Is it true that every graph $G \in \mathcal{G}$ is connected?
(If yes, prove it! If no, construct a disconnected $G \in \mathcal{G}$.) *(10 points)*

d) Is it true that for every graph $G \in \mathcal{G}$, there exist three trails T_1, T_2 and T_3 in G such that every edge of G is contained in exactly one of these three trails?

(20 points)

Recall a trail in G is a walk in G where every edge of G is contained at most once.

2. Let $G = (V, E)$ be a bipartite graph with parts A and B, and let $X \subseteq A$ and $Y \subseteq B$. Suppose there exist two matchings M_X and M_Y in G that cover every vertex in X and Y, respectively. In other words, for each $u \in X$ (resp. $w \in Y$) there is an edge $e \in M_X$ (resp. $f \in M_Y$) such that u (resp. w) is one of the ends of e (resp. f).

Prove that there exists a matching M in G that covers $X \cup Y$. *(50 points)*

3. Let $G = (V, E, w)$ be a weighted complete graph on 11 vertices, where every edge has an integral weight between 100 and 6150, and every spanning tree of G has weight at least 24250. Prove that there exists $v \in V$ such that *(50 points)*

$$\sum_{x \in V} \mathrm{dist}_G(v, x) \geq 15050 \ .$$

<u>**The following story is not important at all for solving this question.**</u>

McGill decided to expand to 10 *other Canadian cities so it has now* 11 *Math departments across the country*[1] *; yay! However, this requires quite some co-ordination in order to have all the midterms at all the departments ready for the exams. This job has been outsourced to a company called Math Assignment Deliveries (MAD).*

MAD is always trying to maximize the profit[2]*. This year, MAD has built a special cable network*[3] *connecting all the campuses and distributed the midterm electronically through the network. Although they tried to be as efficient as possible, they needed* 24250 *kilometers of cables! And since the cable was cheap, it is already rusted and for sure it will not survive the winter.*

Therefore, for the next year McGill changed the policy: all the midterms will be printed in one of the locations (the headquarters), and then physically delivered to the other ones by trucks. This will be again done by MAD, and McGill offers to pay C\$1 *for every kilometer of each of the* 10 *MAD-trucks that will go from the headquarters to the other* 10 *locations. Somewhat surprisingly, MAD got to choose where the headquarters is going to be...*[4]*, and before accepting the offer and doing precise calculations, MAD computed that they will definitely earn at least C*\$15050.

Bonus question for no points: guess what will be the actual profit of MAD.

All the input-data will be available on `http://honza.ucw.cz/m350/`.

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[1]Namely, one in Calgary, Edmonton, Halifax, Montreál, Ottawa, Québec City, Saskatoon, Toronto, Vancouver, Victoria, and Winnipeg. BTW, based on Google Maps, it's about 6150km drive from Halifax to Victoria, and more or less 100km drive between Victoria and Vancouver.

[2]Perhaps like 99.99% of the other companies.

[3]called MAD-work?

[4]... and as it turned out, the "best" choice for the headquarters is Halifax.