MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #1: Paths and Cycles

Due Thursday, September 21st, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. For each of the following statements decide whether it is true or false, and either prove it, or give a counterexample.

a) Let $G = (V, E)$ be a simple graph, and let $\overline{G}$ be the complement of G, i.e., the graph $(V, \overline{E})$, where $\overline{E} := \binom{V}{2} \setminus E$. If G is not connected, then $\overline{G}$ is connected.

Solution: True.

Let C be an arbitrary connected component of G, and let $D := V(G) \setminus C$. Since G is not connected, $D \neq \emptyset$. Fix two vertices $u \in C$ and $v \in D$. It follows that in the graph $\overline{G}$, any vertex in C is connected to any vertex in D by an edge. Moreover, for any two vertices $c_1, c_2 \in C$, there is a path of length two in $\overline{G}$ between c_1 and c_2 via v. Analogously, for any two vertices $d_1, d_2 \in D$, there is a path of length two between d_1 and d_2 via u. So $\overline{G}$ is connected.

b) Let G be a graph and e, f, g be three edges of G. If there is a cycle in G containing e and f, and a cycle containing f and g, then there is a cycle containing e and g.

Solution: True.

Let $e = \{u_1, v_1\}$ and $f = \{u_2, v_2\}$. Fix an arbitrary cycle C_1 containing e and f. Without loss of generality,

$$C_1 = \underbrace{u_2, \ldots, u_1}_{P}, e, \underbrace{v_1, \ldots, v_2}_{Q}, f, u_2$$

where P and Q are the paths on C_1 between u_1 and u_2, and v_1 and v_2, respectively, that both avoid the edges e and f. Clearly, P and Q are vertex-disjoint.

Now, let C_2 be a cycle containing f and g. We may assume C_2 does not contain the edge e, otherwise there is nothing to prove. Define p to be the first vertex on P starting from u_1 that is contained in C_2. Such a vertex must exist, since u_2 is a vertex of C_2 (note it might be that $p = u_1$). Analogously, let q be the first vertex on Q starting from v_1 that is contained in C_2 (again, it might be that $q = v_1$). Let R_1 be the path on C_1 between p and q that contains the edge e. It follows from the construction that $V(R_1) \cap V(C_2) = \{p, q\}$. Now set R_2 to be the path on C_2 between p and q that contains the edge g. The union of the edges of R_1 and R_2 forms a cycle that contains both e and g.

c) Let G be a graph and u, v, w be three vertices of G. If there is a cycle in G containing u and v, and a cycle containing v and w, then there is a cycle containing u and w.

Solution: False; See Figure 1 for a counterexample.

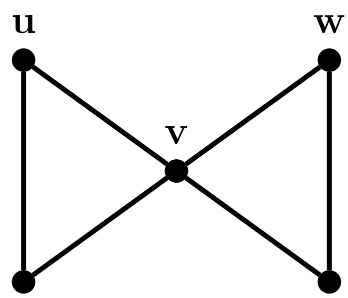

Figure 1: A counterexample for Problem 1b).

d) Let G be a connected graph which contains no path of length more than k. Do every two paths in G of length k have at least one vertex in common?

Solution: True.

Suppose for a contradiction that P_1 and P_2 are two vertex-disjoint paths of length k. Let vertices of P_i, where $1 \leq i \leq 2$, be $v_1^i, v_2^i, \ldots, v_{k+1}^i$, in order. Let Q be a path with one end in $V(P_1)$ and another in $V(P_2)$ chosen to be as short as possible. Let v_n^1 and v_m^2 be the ends of Q, where $1 \leq n, m \leq k+1$. We can suppose without loss of generality that $m, n \geq \lceil k/2+1 \rceil$. Then a path obtained by taking the union of the subpath of P_1 from v_1^1 to v_n^1, the path Q and the subpath of P_2 from v_1^2 to v_m^2 has at least $m+n \geq k+2$ vertices, a contradiction.

2. Let G be a simple graph such that the degree of every vertex is at least k. Prove that G contains a cycle of length at least k. **NOTE that the question was stated a bit ambiguously. Firstly, it should have explicitly mentioned that we assume $k \geq 2$. Secondly, one can actually find a cycle of length at least $k+1$, which was the originally intended statement.**

Solution: Let P be the longest path in G, and let $v \in V(P)$ be one of the ends of P. Since P is the longest path, all the neighbors of v lie somewhere on P (otherwise we could make P longer). Let u be the neighbor of v that has the largest distance from v on P. It follows that the subpath P' of P that goes from u to v contains at least $\deg(v)+1$ vertices (all the neighbors of v, the vertex v itself, and maybe some more vertices). Therefore, the path $P' + \{u, v\}$ is a cycle in G which has the required length.

3. Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \ldots, v_n\}$. Recall that the adjacency matrix A_G of G is a symmetric $n \times n$ matrix with the (i, j)-th entry being

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For every integer k and $i, j \in \{1, 2, \ldots, n\}$, prove that the (i, j)-th entry of $(A_G)^k$ is equal to the number of walks of length k in G from v_i to v_j.

Solution: We prove the statement using induction on k.
For $k = 1$, we observe that the walk of length 1 between two vertices v_i and v_j is equivalent to having an edge $\{v_i, v_j\}$ in G. By definition, A_G has the value 1 exactly at the entries corresponding to such pairs (v_i, v_j).

Suppose the statement holds for all $k' \leq \ell$ (and in particular, for $k' = \ell$), and let $k = \ell + 1$. By the associativity of matrix-multiplication, we know that $(A_G)^k = (A_G)^\ell \times A_G$. On the one hand, by the induction hypothesis, the (x, y)-th entry of $(A_G)^\ell$ is equal to the number of walks of length ℓ in G from v_x to v_y; denote this number by $(A_G)^\ell_{xy}$. On the other, every walk of length k from v_i to v_j can be split into a walk from v_i to v_z of length $\ell = k - 1$ and an edge connecting v_z to v_j. Therefore, the number of walks of length k from v_i to v_j is equal to

$$\sum_{z \in \{1,2,\ldots,n\}} (A_G)^\ell_{iz} \times a_{zj}.$$

This is simply because each walk of length ℓ from v_i to a particular v_z can be extended to a walk of length k if and only if $\{v_z, v_j\} \in E$, i.e., if and only if $a_{zj} = 1$. The definition of matrix multiplication yields that the sum is equal to the (i, j)-th entry of $(A_G)^k$, which finishes the proof.