MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #2: Trees

Due Thursday, September 28st, 8:30AM *Write your answers clearly. Justify all your answers.*

---

1. Prove that a graph $G = (V, E)$ is a tree if and only if G has no cycles and $|V| = |E| + 1$. *(3 points)*
Solution: We already know that every tree T has no cycles (by definition), and satisfies $|V(T)| = |E(T)|+1$ (by Theorem 3.1 from the lecture notes), so it is enough to show that if a graph G has no cycles and satisfies $|V(G)| = |E(G)| + 1$, then G must be a tree. Indeed; the only possibility how G would fail to be a tree is when G is disconnected, i.e., $\text{comp}(G) \geq 2$. However, this contradicts Theorem 3.1 stating that $\text{comp}(G) = |V(G)| - |E(G)|$ for every acyclic graph.

---

2. Let T be a tree. Recall that a leaf of T is a vertex of degree one.

a) Prove that T has exactly two leaves if and only if T is a path on at least two vertices. *(1 point)*

Solution: Clearly, every path on at least two vertices is a tree that has exactly two leaves, so it remains to prove that if a tree T has exactly two leaves, then it is a path. Let u and w be the two leaves of T, and let P be the unique path between u and w in T. We claim that $V(T) = V(P)$, which then indeed implies that T is a path. Suppose for a contradiction that there is a vertex $z \in V(T) \setminus V(P)$. Since T is connected, there is a path Q from u to z. Let x be the last vertex of Q such that the subpath of Q from u to x is also a subpath of P, and let y be the neighbor of x on Q that is not in $V(P)$. Note that x and y are well-defined, because $u \in V(P) \cap V(Q)$ and $z \in V(Q) \setminus V(P)$. However, if $x = u$, then u is not a leaf; a contradiction. If $x \neq u$, then $\deg_T(x) \geq 3$ and Lemma 3.2 states that T has at least 3 leaves; a contradiction.

b) Prove that if T contains a vertex of degree k, then T contains at least k leaves. *(1 point)*

Solution: Let L be the set of leaves in T. Since T is a tree, $|V(T)| = |E(T)|+1$. On the other hand,

$$2|V(T)| - 2 = 2|E(T)| = \sum_{u \in V(T)} \deg(u) = |L| + k + \sum_{\substack{u \in V(T) \setminus L \\ u \neq v}} \deg(u).$$

Since every vertex $u \in V \setminus L$ has degree at least 2, it follows that

$$\sum_{\substack{u \in V(T) \setminus L \\ u \neq v}} \deg(u) \geq 2(|V(T)| - |L| - 1) = 2|V(T)| - 2|L| - 2.$$

Combining the two derivations together, we conclude that

$$2|V(T)| - 2 \geq |L| + k + 2|V(T)| - 2|L| - 2 = k - |L| + 2|V(T)| - 2,$$

which after rearranging the terms yields $|L| \geq k$.

---

3. Let K_n^- be the graph with the vertex-set $\{1, 2, \ldots, n\}$ and the edge-set $\binom{[n]}{2} - \{1, 2\}$. Find a simple closed-form formula for the number of spanning trees of K_n^-, and prove it is correct. *(5 points)*

Solution: Fix an integer $n \geq 2$. For two integers x and y such that $1 \leq x < y \leq n$, we define s_{xy} to be the number of spanning trees of K_n that contains the edge $\{x, y\}$. We claim that $s_{12} = 2n^{n-3}$, and hence the number of spanning trees of K_n^- is equal to

$$n^{n-2} - 2n^{n-3} = n^{n-3} \cdot (n - 2).$$

Firstly, let us observe that, by the symmetry of the complete graph, we have $s_{12} = s_{ab}$ for all $1 \leq a < b \leq n$. Fix one such a and b. Let S_{12} be the set of spanning trees of K_n containing the edge $\{1, 2\}$, and analogously S_{ab} the set of spanning trees containing the edge $\{a, b\}$. So we want to prove that $s_{12} = |S_{12}| = S|S_{ab}| = s_{ab}$, and we prove this by constructing a bijection $f : S_{12} \to S_{ab}$ defined in the following way:

- if $T \in S_{12}$ is a spanning tree that contains also the edge $\{a, b\}$, then $f(T) = T$,
- otherwise let $f(T)$ be the tree obtained from T by relabelling the vertex 1 to a, the vertex 2 to b, the vertex a to 1, and the vertex b to 2.

It readily follows that f is a bijection from S_{12} to S_{ab}, so, indeed $s_{12} = s_{ab}$.
Ok, let's now count in two different ways the number of pairs (T, e) where T is a spanning tree of K_n and $e \in E(T)$. On the one hand, K_n has n^{n-2} spanning trees and every spanning tree has $(n-1)$ edges. On the other, there are $\binom{n}{2}$ edges in K_n and each of them is contained in exactly s_{12} spanning trees. Therefore,

$$n^{n-2} \cdot (n-1) = \binom{n}{2} \cdot s_{12} = \frac{n(n-1)}{2} \cdot s_{12}.$$

Rearranging the terms yields $s_{12} = 2n^{n-3}$ as we claimed.