MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #3: Components, Minimum Spanning Trees, Shortest Paths

Due Thursday, October 5th, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. Determine what is the maximum number of edges of an n-vertex graph G with k connected components (i.e., $\text{comp}(G) = k$). *(2 points)*

Solution: We claim that the best is to split the vertex-set into $(k-1)$ isolated vertices and one big connected component with $n-k+1$ vertices which induces a complete graph K_{n-k+1}. Suppose for a contradiction that the maximum is achieved by a graph $G = (V, E)$ with $\text{comp}(G) = k$ that has (at least) two components, say V_1 and V_2, with $|V_1| \geq |V_2| \geq 2$. Then, pick $v \in V_2$ arbitrarily and consider the graph G' obtained by removing v from G and adding a new vertex v' adjacent to all the vertices in V_1. The degree of v in G was at most $|V_2| - 1$, however, the degree of v' in G' is $|V_1|$. Therefore,

$$|E(G')| - |E(G)| \geq |V_1| - |V_2| + 1 \geq 1;$$

a contradiction. Hence the maximum number of edges is $\binom{n-k+1}{2}$.

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2. Let V be a finite set, and let $d : V \times V \to \mathbb{N} \cup \{0\}$ be any non-negative integral function that satisfies:

1) $d(u, w) = 0 \iff u = w$,
2) $d(u, w) = d(w, u)$ for every $u, w \in V$,
3) $d(u, w) \leq d(u, v) + d(v, w)$ for every $u, v, w \in V$, and
4) if $d(u, w) > 1$, then there exists $v \in V \setminus \{u, w\}$ such that $d(u, w) = d(u, v) + d(v, w)$.

Prove that there exists a simple graph $G = (V, E)$ such that $\text{dist}_G = d$. *(4 points)*

Solution: For a given function d satisfying the properties (1)-(4), we will construct a graph $G = (V, E)$ with the edge-set $E = \{uw \,|\, d(u, w) = 1\}$. What remains to prove is that $\text{dist}_G(u, w) = d(u, w)$ for every $u, w \in V$.
Firstly, we observe that $d(u, w) \leq \text{dist}_G(u, w)$. Indeed, let $P = v_0, e_1, v_1, \ldots, e_\ell, v_\ell$ be a shortest path in G between $u = v_0$ and $w = v_\ell$. By the property (3),

$$d(u, w) \leq \sum_{i=1}^{\ell} d(v_{i-1}, v_i) = \sum_{i=1}^{\ell} 1 = \ell = \text{dist}_G(u, w).$$

Now suppose for a contradiction that there is a pair of vertices $u, w \in V$ such that $\text{dist}_G(u, w) > d(u, w)$. Among all such pairs, we choose the one that has the value $d(u, w)$ as small as possible. Clearly, $d(u, w) \geq 2$. Using the property (4), let $v \in V \setminus \{u, w\}$ be some vertex that satisfies $d(u, w) = d(u, v) + d(v, w)$. Since $v \neq u$ and $v \neq w$, the property (1) ensures that both $d(v, u) \geq 1$ and $d(v, w) \geq 1$, which in turn implies that $d(u, v) < d(u, w)$ as well as $d(v, w) < d(u, w)$. Therefore, by the choice of the vertices u and w, we have $\text{dist}_G(u, v) = d(u, v)$ and $\text{dist}_G(v, w) = d(v, w)$. Putting the pieces together, we conclude that

$$\text{dist}_G(u, w) > d(u, w) = d(u, v) + d(v, w) = \text{dist}_G(u, v) + \text{dist}_G(v, w)\,;$$

a contradiction.

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3. Let $G = (V, E)$ be a simple graph. A set $A \subseteq V$ is called *an independent set in* G if the induced subgraph $G[A]$ contains no edge, i.e., if every edge $e \in E$ is incident to at most one vertex in A. Define $\alpha(G)$ to be the maximum cardinality of an independent set in G.
Suppose $G = (V, E)$ is a simple graph. Prove that there is an integer $\ell \leq \alpha(G)$ and a collection of paths $P_1, P_2, \ldots, P_\ell$ such that for every vertex $v \in V$ there exists *exactly one* path P_i, where $i \in \{1, \ldots, \ell\}$, with $v \in V(P_i)$. *(4 points)*

Solution: We try to follow the hint. Firstly, we prove that for some integer ℓ there exists a collection of paths $P_1, \ldots, P_\ell$ with the property that every vertex $v \in V$ is in exactly one P_i. Indeed, for $\ell = |V|$ the task is actually trivial; just consider $|V|$ one-vertex paths, each containing a different vertex of G.
Now let ℓ_0 be the minimum ℓ such that there exists a collection of paths $P_1, \ldots, P_\ell$ with the property that every vertex $v \in V$ is in exactly one P_i. Suppose $\ell_0 > \alpha(G)$, and let $P_1, \ldots, P_{\ell_0}$ be the paths. For every $i \in \{1, \ldots, \ell_0\}$, let u_i be one of the two end-vertices of the path P_i. Set $U := \{u_1, u_2, \ldots, u_{\ell_0}\}$. Since the paths $P_1, \ldots, P_{\ell_0}$ were disjoint, we have $|U| = \ell_0 > \alpha(G)$. But this means U cannot be independent, so it contains an edge $\{u_i, u_j\}$ for two distinct $i, j \in \{1, \ldots, \ell_0\}$. Let Q be the graph with the vertex set $V(P_i) \cup V(P_j)$ and the edge set $E(P_i) \cup E(P_j) \cup \{u_i, u_j\}$. It follows that Q is a path. But then replacing the paths P_i and P_j with Q yields a collection of $\ell_0 - 1$ paths such that every vertex $v \in V$ is in exactly one of them; a contradiction with the minimality of ℓ_0.