MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #4: Eulerian Tours, Bipartite graphs

Due Thursday, October 12th, 8:30AM *Write your answers clearly. Justify all your answers.*

---

1. Let G be a connected multigraph. We say that $F \subseteq E(G)$ is *even-degree*, if every vertex of G is incident with an even number of edges in F. Let T be an arbitrary spanning tree of G. Prove that there exists an even-degree set $F_T \subseteq E(G)$ such that $F_T \cup E(T) = E(G)$. *(3 points)*

Solution: We claim that if F_1 and F_2 are both even-degree then so is $F_1 \triangle F_2 := (F_1 - F_2) \cup (F_2 - F_1)$. Indeed, if E_1 and E_2 are the sets of edges in F_1 and F_2, respectively, incident to the vertex v, then $|E_1 \triangle E_2| = |E_1| + |E_2| - 2|E_1 \cap E_2|$, which is even if $|E_1|$ and $|E_2|$ are even.
For $e \in E(G) \setminus E(T)$, let $F(e)$ be the edge set of the fundamental cycle of e w.r.t. T, i.e., the cycle formed by e and the unique path between the ends of e in T. Clearly, $F(e)$ is even-degree. Let

$$F_T := F(e_1) \triangle F(e_2) \dots \triangle F(e_k),$$

where $E(G) \setminus E(T) = \{e_1, e_2, \dots, e_k\}$. Then F is an even-degree set, by the claim above, and $F \cup E(T) = E(G)$, as $e_i \in F(e_i)$ and $e_i \notin F(e_j)$ for $i, j \in \{1, 2, \dots, k\}$, $i \neq j$.

---

2. Let $G = (V, E)$ be a connected multigraph that has exactly 2ℓ vertices of odd degree. Prove that there exists a collection of trails $T_1, T_2, \dots T_\ell$ in G such that every edge $e \in E$ is in exactly one trail T_i, where $i \in \{1, \dots, \ell\}$. *(4 points)*

Solution: Let $v_1, v_2, \dots, v_{2\ell} \in V$ be the vertices of G that have odd degrees. We construct a multigraph G' by adding to the graph G exactly ℓ new edges $e_1, \dots, e_\ell$, where the i-th edge connects the vertices v_{2i-1} and v_{2i}. It readily follows that G' is a connected multigraph and the degree of every vertex of G' is even. Therefore, G' contains an Eulerian tour

$$T = v_0 f_1 v_1 f_2 v_2 \dots f_{m'} v_{m'} \ ,$$

where m' is the number of edges of G'.
Let $k_1, \dots, k_\ell$ be the indices of the edges $e_1, \dots, e_\ell$ in the tour T. Without loss of generality, $k_\ell = m'$, and for convenience, we define $k_0 := 0$. For every $i \in \{1, \dots, \ell\}$, let

$$T_i := v_{k_{i-1}} e_{k_{i-1}+1} v_{k_{i-1}+1} \dots v_{k_i - 1}.$$

It follows that each T_i is a trail in G and every edge $e \in E$ is contained in exactly one of these trails.

---

3. Prove that every loopless graph $G = (V, E)$ contains a subgraph H that is bipartite and $\deg_H(v) \geq \lceil \deg_G(v)/2 \rceil$ for every $v \in V$. In particular, H has at least $\lceil |E|/2 \rceil$ edges. *(3 points)*

Solution: Let H be a bipartite subgraph of G that has maximum number of edges, and let $A \subseteq V$ and $B \subseteq V$ be the parts of the bipartition. Clearly, H contains every edge of G that has exactly one end in A (and hence also exactly one end in B). We claim that $\deg_H(v) \geq \frac{1}{2}\deg_G(v)$ for every $v \in V$.
Suppose for contradiction there exists a vertex $v \in V$ such that $\deg_H(v) < \frac{1}{2}\deg_G(v)$. Without loss of generality, $v \in A$. But then the vertex v has (in G) more neighbors inside A than inside B. Therefore, the bipartite subgraph H' containing all the edges of G that goes between the parts $A' := A \setminus \{v\}$ and $B' := B \cup \{v\}$ contains strictly more edges than H; a contradiction.