1. Let $T$ be a tree. Prove that $T$ has a perfect matching if and only if for every vertex $v \in V(T)$ the subgraph $T - v$ contains exactly one connected component with odd number of vertices. (2 points)

**Solution:** Clearly, if $T$ has a perfect matching then it satisfies Tutte’s condition, and, in particular, for every $v \in V$, the subgraph $T - v$ has exactly one connected component with odd number of vertices. Now we show that if $\text{odd}_T(V - v) = 1$ for all $v \in V$ then $T$ has a perfect matching. Firstly, if $v$ is a leaf then $T - v$ has just one connected component, and hence $T$ has even number of vertices. We now proceed by proving the statement by induction on $n$. If $n = 2$, then $T$ is a single edge which also forms a perfect matching in $T$. Let us now assume $n \geq 4$. Let $u$ be a leaf of $T$ and $w$ its unique neighbor. Consider $T' := T - u - w$; we show that $T'$ has a perfect matching $M'$ which together with the edge $\{u, w\}$ forms a perfect matching in $T$.

Suppose for contradiction $T'$ has no perfect matching. By the induction hypothesis, there is a vertex $v \in V(T') = V \setminus \{u, v\}$ such that $T' - v$ has $\ell \geq 3$ connected components $C_1, C_2, \ldots, C_{\ell}$ of odd size. However, since $T - v$ has only one odd component, the vertex $w$ must have a neighbor in at least two of the components $C_i$ and $C_j$ (where $i \neq j$). Therefore, there are at least two different paths from $v$ to $w$ (one path passes through $C_i$, the other one through $C_j$); a contradiction.

2. Let $G$ be a 3-regular simple graph with no cut-edge, and let $e \in E(G)$ be an edge of $G$.

a) Prove that $G$ has a perfect matching $M_1$ such that $e \in M_1$. (2 points)

**Solution:** Let $u$ and $w$ be the two endpoints of $e$, and let $H := G - u - w$. It is enough to show that $H$ has a perfect matching $M'$, since $M' + e$ will be a perfect matching of $G$ that contains $e$.

Let $V := V(G)$ and $W := V(H) = V \setminus \{u, w\}$. Suppose for contradiction $H$ does not have a perfect matching. By Tutte’s theorem, there exists $S_0 \subseteq W$ such that $\text{odd}_H(S_0) > |S_0|$, where $S_0 = W \setminus S_0$. First, we observe that the parity of $\text{odd}_H(S_0)$ and $|S_0|$ is the same. Indeed, recall that $|V|$ is even and that

$$|V| - 2 = |W| = \sum_{c \text{ even component of } H[S_0]} |C| + \sum_{c \text{ odd component of } H[S_0]} |C| + |S_0|.$$ 

Therefore, $\text{odd}_H(S_0) \geq |S_0| + 2$, and for $S := S_0 \cup \{u, w\}$ we have

$$\text{odd}_G(V \setminus S) = \text{odd}_H(S_0) \geq |S_0| + 2 = |S|.$$ 

Now we look closer to the situation in $G$ and the set of vertices $S$. The number of edges between $S$ and $V \setminus S$ is at most $3(|S| - 2) + 4 = 3|S| - 2$ because $u$ is adjacent to at most two vertices in $V \setminus S$ and the same holds also for $v$. On the other hand, there are at least $|S|$ odd components in $G(V \setminus S)$. As in the proof of Petersen’s theorem in the lecture, each such odd connected component must receive at least 3 edges from the vertices in $S$ (only one edge would mean a cut-edge in $G$, only two edges violates the parity constraint). So the number of edges between $S$ and $V \setminus S$ must be at least $3|S|$; a contradiction.

b) Prove that $G$ has a perfect matching $M_2$ such that $e \notin M_2$. (2 points)

**Solution:** This immediately follows from (a). Let $v$ be one of the endpoints of $e$ and let $f$ be one of the other two edges incident to $v$ (chosen arbitrarily). A perfect matching $M$ containing $f$ guaranteed by (a) clearly cannot contain $e$.
3.

a) Let \( k \geq 3 \) and \( G = (V, E) \) a \( k \)-regular connected graph with even number of vertices. Suppose \( G \) has the property that for every set of edges \( F \subseteq E \) of size \( k-2 \), the subgraph \((V, E \setminus F)\) is still connected (graphs with this property are called \((k-1)\)-edge-connected). Prove that \( G \) has a perfect matching.

**Solution:** Fix the value of \( k \), and a \( k \)-regular \((k-1)\)-edge-connected graph \( G = (V, E) \). Suppose \( G \) has no perfect matching. Then by Tutte’s theorem, there is \( S \subseteq V \) with \( |S| = \ell \) and the odd connected components of \( G-S \) are \( C_1, C_2, \ldots, C_m \) with \( m > \ell \).

**Claim.** Every odd connected component \( C_i \) sends at least \( k \) edges to \( S \).

**Proof.** By the connectivity assumption on \( G \), the only situation when this claim would be false is when \( C_i \) sends exactly \( k-1 \) edges out. Let \( H_i \) be the induced subgraph of \( G \) defined by \( C_i \). By definition,

\[
\sum_{v \in C_i} \deg_{H_i}(v) = k \cdot |C_i| - (k-1) = k \cdot (|C_i| - 1) + 1.
\]

Since \( |C_i| - 1 \) is even, the above sum must be odd, which contradicts the hand-shaking lemma. \( \square \)

Let’s count the edges of \( G \) between \( S \) and \( \bigcup_{i \in \{1, \ldots, m\}} C_i \) now. On the one hand, that number is at most \( \ell \cdot k \). On the other hand, the claim above yields the number of such edges is at least \( m \cdot k \), and hence \( m \leq \ell \); a contradiction.

b) For every \( k \geq 3 \), construct a \( k \)-regular graph \( Z_k = (V, E) \) with even number of vertices which has the property that for every subset of edges \( F \subseteq E \) of size \( k-3 \), the subgraph \((V, E \setminus F)\) is still connected (i.e., \( Z_k \) is \((k-2)\)-edge-connected), but yet \( Z_k \) has no perfect matching. \( \hfill (2 \text{ points}) \)

**Solution:** This was perhaps a bit harder exercise than I originally thought; I apologize.

Firstly, let us show that there exists a \((k-2)\)-edge-connected graph where \((k+1)\) vertices have degree \( k \), and \((k-2)\) vertices have degree \((k-1)\).

**Lemma.** Fix an integer \( k \geq 3 \). There exist a \((2k-1)\)-vertex \((k-2)\)-edge-connected simple graph \( H_k = (V_k, E_k) \) with \( V_k = \{y_1, y_2, \ldots, y_{k+1}, z_1, z_2, \ldots, z_{k-2}\} \), where all the vertices \( y_i \), for \( 1 \leq i \leq k+1 \), have degree exactly \( k \), and all the vertices \( z_j \), for \( 1 \leq j \leq k-2 \), have degree exactly \( k-1 \).

**Proof.** Start with a \( k \)-vertex complete graph on the vertices \( \{y_1, y_2, \ldots, y_k\} \), plus a \((k-2)\)-vertex complete graph on the vertices \( \{z_1, z_2, \ldots, z_{k-2}\} \). Next, place an edge from the vertex \( y_{k+1} \) to the vertices \( y_{k-1} \) and \( y_k \), then \((k-2)\) edges of the form \( y_j z_j \), where \( 1 \leq j \leq k-2 \), and also \((k-2)\) edges of the form \( y_{k+1} z_j \), again where \( 1 \leq j \leq k-2 \). Observe that every vertex \( y_i \), where \( 1 \leq i \leq k+1 \), has degree exactly \( k \), and every vertex \( z_j \), where \( 1 \leq j \leq k-2 \), has degree exactly \( k-1 \).

It remains to show that \( H_k \) is \((k-2)\)-edge-connected. In other words, we want to show that for any two vertices \( u, w \in V_k \) and any \((k-3)\) edges \( F \subseteq E_k \), there will still be some path from \( u \) to \( w \) in the subgraph \( H_k - F = (V_k, E_k \setminus F) \). This will definitely be true if we find \((k-2)\) edge-disjoint paths between \( u \) and \( w \) in the graph \( H_k \), simply because removing \( F \) can break at most \((k-3)\) of them.

**Claim.** For every \( u, w \in V_k \), there exist \((k-2)\) edge-disjoint paths from \( u \) to \( w \) in \( H_k \).

Unfortunately, we need to go through different cases here...
Case 1) $u, w \in \{y_1, y_2, \ldots, y_k\}$. Indeed; actually, we can easily find even $(k-1)$ edge-disjoint paths.
Without loss of generality, $u = y_1$ and $w = y_2$. Then one path is simply the edge $y_1, y_2$, and then there $(k-2)$ paths of the form $u = y_1, y_i, y_2 = w$, where $3 \leq i \leq k$.

Case 2) $u \in \{y_1, y_2, \ldots, y_{k-1}\}$ and $w = y_{k+1}$. By symmetry, we may assume $u = y_1$. We actually describe $k$ edge-disjoint paths this time: there are three paths $P_1 = y_1, z_1, y_{k+1}, P_2 = y_1, y_{k-1}, y_{k+1}, P_3 = y_1, y_k, y_{k+1}$, and, for every $i$ such that $2 \leq i \leq k-2$, there is a path $P_{i+2} = y_1, y_i, z_i, y_{k+1}$.

Case 3) $u \in \{y_{k-1}, y_k\}$ and $w = y_{k+1}$. Again by symmetry, we may assume $u = y_k$. There are $(k-2)$-edge-disjoint paths of the form $y_k y_i z_i y_{k+1}$ (plus also an edge $y_k y_{k+1}$).

Case 4) $u \in \{y_1, y_2, \ldots, y_{k-2}\}$ and $w \in \{z_1, \ldots, z_{k-2}\}$. Without loss of generality, $u = y_1$. If $w = z_1$, then there is an edge $y_1, z_1$ and $(k-3)$ edge-disjoint paths of the form $y_1, y_i, z_i, z_1$, where $2 \leq i \leq k-2$. If $w \neq z_1$, then by symmetry we may assume $w = z_2$. This time, we have two paths $y_1, z_1, z_2$ and $y_1, y_2, z_2$, plus $(k-4)$ paths of the form $y_1, yi, z_i, z_2$, where $3 \leq i \leq k-2$.

Case 5) $u \in \{y_{k-1}, y_k\}$ and $w \in \{z_1, \ldots, z_{k-2}\}$. Without loss of generality, $u = y_k$ and $w = z_1$. The sought paths are $P_i = y_k, y_i, z_i$ and for every $i \leq k-2$, we take $P_i = y_{k+1}, z_i, z_1$.

Case 6) $u = y_{k+1}$ and $w \in \{z_1, \ldots, z_{k-2}\}$. By symmetry, we assume $w = z_1$. This time, we take $P_1 = y_{k+1}, z_1$ and for $2 \leq i \leq k-2$, we take $P_i = y_{k+1}, z_i, z_1$.

Case 7) $u, w \in \{z_1, \ldots, z_{k-2}\}$. Finally, the last case! Without loss of generality, $u = z_1$ and $v = z_2$.
This time, we take $P_1 = z_1, z_2, P_2 = z_1, y_{k+1}, z_2$, and for $3 \leq i \leq k-2$, we take $P_i = z_1, z_i, z_2$.

This finishes the proof of the claim as well as the proof of the lemma; Yay!

Now we claim that the following graph $Z_k$ is both $(k-2)$-edge-connected and yet has no perfect matching: take $k$ vertex-disjoint copies $C^1, C^2, \ldots, C^k$ of the graph $H_k$, add $(k-2)$ new vertices $v_1, v_2, \ldots, v_{k-2}$, and for every $i \in \{1, \ldots, k-2\}$ and $j \in \{1, \ldots, k\}$ connect $v_i$ to $z_j$, where $z_j$ is the $i$-th vertex of degree $(k-1)$ in the copy $C^j$.

Clearly, $Z_k$ is $k$-regular and has no perfect matching. The latter one is because

$$\text{odd} G(V(Z_k) - S) = k = |S| + 2 \quad \text{for } S = \{v_1, \ldots, v_{k-2}\}.$$  

It remains to show that $Z_k$ stays connected even after removing arbitrary $(k-3)$ edges $F \subseteq E(Z_k)$. Suppose there is $F \subseteq E(Z_k)$ of size $(k-3)$ such that $G - F$ is not connected. Since $H_k$ is $(k-2)$-edge-connected, we may assume that every $f \in F$ is incident to some vertex $v_i$. This reduces the problem of $(k-2)$-edge-connectivity of the graph $Z_k$ to the following simple lemma:

Lemma. For every $k \geq 3$, the complete bipartite graph $K_{k,k-2}$ is $(k-2)$-edge-connected.

Proof. Let $A = \{a_1, \ldots, a_k\}$ be the part of size $k$ and $B = \{b_1, \ldots, b_{k-2}\}$ be the part of size $(k-2)$. We will show that between any two vertices $u, w \in A \cup B$, there are $(k-2)$-edge-disjoint paths. Again, we go over three cases here:

Case 1) $u, w \in A$. The sought paths are $P_i = u, b_i, w$, where $1 \leq i \leq k-2$.

Case 2) $u \in A$ and $w \in B$. By symmetry, we may assume $u = a_1$. Let $w = b_j$ for some $j \in \{1, \ldots, k-2\}$. Apart from the edge $a_1, b_j$, there are also $(k-3)$ paths of the form $a_1, b_i, a_i, b_j$, where $1 \leq i \leq k-2$ and $i \neq j$.

Case 3) $u, w \in B$. This time, we find $k$ edge-disjoint paths $P_i = u, a_i, w$, where $1 \leq i \leq k$.

This finishes the proof of the lemma, which in turn yields that $Z_k$ is indeed $(k-2)$-edge-connected.
Figure 1: The graph $Z_3$ for Question 3b.