MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #7: Ramsey Theory

Due Thursday, November 2nd, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. For given integers k, ℓ and m, recall that $R(k, \ell, m)$ is the smallest integer N such that any red/blue/green coloring of $E(K_N)$ contains at least one of the following subgraphs: a red copy of K_k, a blue copy of K_ℓ, or a green copy of K_m. Prove that

$$R(k, \ell, m) \leq \frac{(k + \ell + m - 3)!}{(k-1)!(\ell-1)!(m-1)!} \; . \qquad \textit{(3 points)}$$

Solution:
Firstly, it is convenient to observe that the claimed upper-bound is a multinomial coefficient, i.e.,

$$\frac{(k + \ell + m - 3)!}{(k-1)!(\ell-1)!(m-1)!} = \binom{(k-1) + (\ell-1) + (m-1)}{k-1,\ \ell-1,\ m-1} = \binom{k+\ell+m-3}{k-1,\ \ell-1,\ m-1}.$$

Moreover, either a simple combinatorial argument or a tedious but straightforward calculation shows that

$$\binom{a+b+c}{a,\ b,\ c} = \binom{(a-1)+b+c}{a-1,\ b,\ c} + \binom{a+(b-1)+c}{a,\ b-1,\ c} + \binom{a+b+(c-1)}{a,\ b,\ c-1} \quad \text{for every } a, b, c \in \mathbb{N}.$$

We prove the statement by induction on $k + \ell + m$. Without loss of generality, we may assume $k \geq \ell \geq m$. If $m = 1$, then the statement is clearly true, so we may assume $m \geq 2$ (and hence $k + \ell + m \geq 6$). We set $n := \binom{k+\ell+m-3}{k-1,\, \ell-1,\, m-1}$, and consider any red/blue/green edge-coloring of $E(K_n)$. Let $v \in V(K_n)$, and R the red neighbors of v, B the blue neighbors of v and G the green neighbors of v. If for all the three sets R, B and G we would have

a) $|R| \leq \binom{(k-2)+(\ell-1)+(m-1)}{k-2,\, \ell-1,\, m-1} - 1$,

b) $|B| \leq \binom{(k-1)+(\ell-2)+(m-1)}{k-1,\, \ell-2,\, m-1} - 1$, and

c) $|G| \leq \binom{(k-1)+(\ell-1)+(m-2)}{k-1,\, \ell-1,\, m-2} - 1$,

then by the above identity for multinomial coefficients we get

$$n = 1 + |A| + |B| + |C| \leq \binom{k+\ell+m-3}{k-1,\ \ell-1,\ m-1} - 2,$$

which is impossible. Therefore, it must be that

a) $|R| \geq \binom{(k-2)+(\ell-1)+(m-1)}{k-2,\, \ell-1,\, m-1}$, or

b) $|B| \geq \binom{(k-1)+(\ell-2)+(m-1)}{k-1,\, \ell-2,\, m-1}$, or

c) $|G| \geq \binom{(k-1)+(\ell-1)+(m-2)}{k-1,\, \ell-1,\, m-2}$.

Without loss of generality, we may assume that the option (a) happened. But then by the induction hypothesis, the subgraph induced by A either contains a red K_{k-1}, a blue K_ℓ, or a green K_m. Therefore, we take the blue K_ℓ or the green K_m, or we extend the red K_{k-1} by v into a red K_k, respectively.

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2. Let $R_k(3) := R(\overbrace{3, 3, \ldots, 3}^{k})$ is the minimum integer n such that any k-coloring of $E(K_n)$ contains a monchromatic K_3. Prove that $R_k(3) \leq 3k!$ for any integer $k \geq 1$. *(3 points)*

Solution:
We proceed by induction on k. Clearly, the formula holds for $k = 1$ and $k = 2$ as well ($R(3, 3) = 6$). Suppose $k > 2$ and fix any k-coloring of $E(K_{3k!})$. Let v be a vertex and $i \in \{1, \ldots, k\}$ be the most frequent color on the edges incident to v. Without loss of generality, $i = k$. Set N to be the set of vertices u such that $\{u, v\}$ has color k. We claim that $|N| \geq 3(k-1)!$ as otherwise the total number of vertices in $K_{3k!}$ would be at most

$$k \cdot ((3k-1)! - 1) + 1 = 3k! - (k-1) < 3k!.$$

Now either at least one edge with both endpoints in N has color k, in which case we are done, or, we can apply induction on the $(k-1)$-coloring of $K_{|N|}$ that is induced by the coloring of the edges inside N. Since $|N| \geq 3(k-1)!$, the induction hypothesis yields a monochromatic triangle inside N.

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3. A *torunament* is an oriented graph where every two vertices u and v are joined by either an oriented edge from u to v ($u \to v$), or from v to u. More formally, it is a triple (V, E, o) where V is a set of vertices, $E = \binom{V}{2}$ a set of edges, and $o : E \to V$ such that $o(\{u, v\}) \in \{u, v\}$ a function determining the orientation by letting $o(e)$ to be the source of the edge $e \in E$.
An *oriented cycle* in a tournament $T = (V, E, o)$ is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ where $v_i \in V$, $e_i \in E$, $v_1 = v_{k+1}$, $e_i = \{v_i, v_{i+1}\}$ and $o(e_i) = v_i$ for all $i \in \{1, 2, \ldots, k\}$. A vertex-subset U of V is called *acyclic set* in T if the subtournament of T induced by U contains no oriented cycle.

Let T be an n-vertex tournament. Prove that T contains an acyclic set of size $\lfloor \log_2(n) \rfloor + 1$. *(4 points)*

Solution: Without loss of generality, we may assume that $n = 2^t$ for some $t \in \mathbb{N}$ as otherwise we can remove arbitrary $n - 2^{\lfloor \log_2(n) \rfloor}$ vertices from T.
We prove the statement by induction on t. If $t = 0$, i.e., $n = 1$, then the single vertex of T indeed forms an acyclic set. Suppose that $t > 0$. Let $v \in V(T)$ be an arbitrary vertex of T, and let N_v^+ and N_v^- be the set of out-neighbors and in-neighbors of v, respectively. We claim that either $|N_v^+|$ or $|N_v^-|$ have size at least 2^{t-1}. Indeed, as otherwise

$$n \leq 2 \cdot \left(2^{t-1} - 1\right) + 1 = 2^t - 1 \ ,$$

a contradiction. Without loss of generality, $|N_v^+| \geq 2^{t-1}$, and let T^+ be the subtournament of T defined by the first 2^{t-1} vertices of $|N_v^+|$. By induction hypothesis, T^+ contains an acyclic set of size t, which together with the vertex v forms an acyclic set of size $t + 1$.