MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #8: Connectivity & Menger's theorem, Network flows

Due Tuesday, November 14th, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. Let $G = (V, E)$ be a 2-connected graph and $v \in V$ a vertex of G. Prove that there exists a vertex $u \in V$ such that $\{u, v\} \in E$ and the graph $G - u - v$ is connected. *(3 points)*

Solution: Let U be the set of neighbors of v in G. Let T be a connected subgraph of $G - v$ with the minimum number of edges such that $U \subseteq V(T)$. It is easy to see that T is a tree, and that every leaf of T is a neighbor of v. Let u be a leaf of T. Then $T - u$ is connected. Suppose for a contradiction that $G - u - v$ is not connected and consider a component C of $G - u - v$ which does not contain $T - u$. Thus C contains no neighbor of v and so it is a connected component of $G - u$ different from $T - u$. Therefore, $G - u$ is not connected, contradicting 2-connectivity of G.

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2. Let $H = (V, E)$ be a graph and let $U \subseteq V$. We define $H \oplus_U \{v\}$ to be the graph obtained from H by adding a new vertex v, which is then joined to every vertex in U. In other words,

$$H \oplus_U \{v\} = (V \cup \{v\}, E \cup \{\{u, v\} : u \in U\}) .$$

a) Prove that if $G = (V, E)$ is a k-connected graph and $U \subseteq V$ has size k, then the graph $G \oplus_U \{v\}$ is k-connected as well. *(2 points)*

Solution: Suppose for a contradiction $G' := G \oplus_U \{v\}$ is not k-connected. By Menger's theorem, there exists a vertex cut $S \subseteq V(G')$ of size at most $k - 1$. Clearly, if $v \in S$, then $G' - S$ is actually a subgraph of G with at least $|V| - k - 2$ vertices, which is definitely connected (in fact, it is even 2-connected) by the connectivity assumption on G.

Now consider $v \notin S$. Let C_1 and C_2 be different connected components of $G' - S$. We claim that both C_1 and C_2 contain a vertex from the set V. If not, then one of the components, say C_1, would contain only the vertex v. However, since $|U| = k$, there is at least one vertex $u \in U \setminus S$, and this vertex must be in C_1 as well; a contradiction.

Let $u_1 \in V(C_1) \cap V$ and $u_2 \in V(C_2) \cap V$. It follows that every path in G between u_1 and u_2 have to pass through the set S, which is a contradiction with G being k-connected.

b) Let $G = (V, E)$ be a k-connected graph and $U, W \subseteq V$ two vertex-subsets, each of size k. Prove that there exist k pairwise vertex-disjoint paths $P_1, \dots P_k$ such that for every $i \in \{1, \dots, k\}$, the path P_i have one endpoint in U and the other endpoint in W. *(1 point)*

Solution: Let $G' := (G \oplus_U u) \oplus_W w$. By the part (a), G' is k-connected. Therefore, G' contains k internally disjoint paths $Q_1, \dots, Q_k$ between u and w. For every $i \in \{1, \dots, k\}$, let $P_i := Q_i - u - w$. It follows that these are k vertex-disjoint paths in G, each with exactly one end in U and the other in W.

c) Let $G = (V, E)$ be a 2-connected graph. Show that for any triple of distinct vertices $u, v, w \in V$ there is a path in G from u to v passing through w, i.e., w is an inner-vertex of the path. *(1 point)*

Solution: Let $G' := G \oplus_U z$ for $U := \{u, v\}$. Again, the part (a) yields that G' is 2-connected. Hence G' contains 2 internally vertex-disjoint paths Q_1 and Q_2 between z and w. Taking their union and removing the vertex z yields the desired path between u and v that passes through w.

3. Let $G = (V, E)$ be a directed graph (digraph) and for each edge $e \in E$, let $\phi(e) \geq 0$ be a non-negative integer. Show that if for every vertex v

$$\sum_{e \in \partial^-(v)} \phi(e) = \sum_{e \in \partial^+(v)} \phi(e) \ ,$$

then there is a collection of directed cycles $C_1, ..., C_k$ (possibly with repetition) so that for every edge e of G, it holds that $|\{i : 1 \leq i \leq k,\ e \in E(C_i)\}| = \phi(e)$. *(3 points)*

Solution: Induction on $S := \sum_{e \in E(G)} \phi(e)$. Base case: $S = 0$ is trivial. For the induction step, it suffices to find a directed cycle C in G so that $\phi(e) \geq 1$ for every edge $e \in E(G)$, as one can then apply the induction hypothesis to

$$\phi'(e) := \begin{cases} \phi(e), & \text{if } e \notin E(C) \\ \phi(e) - 1, & \text{if } e \in E(C) \end{cases}$$

Let e be an edge of G with $\phi(e) \geq 1$, a tail u and a head v. Then ϕ restricted to $E(G) - e$ is a v-u-flow of value 1. By Lemma 11.3 from the lecture notes, there exists a directed path P in $G - e$ so that ϕ is positive on every edge of the path. The path P together with e forms the desired cycle.