MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #9: Proper vertex-colorings of graphs

Due Tuesday, November 21st, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. Let $G = (V, E)$ be a simple graph and $\overline{G} = \left(V, \binom{V}{2} - E\right)$ its complement. Prove that $\chi(G)\cdot\chi(\overline{G}) \geq |V|$.
(3 points)

Solution: Let us combine the two standard estimates on the chromatic number we know:

- $\chi(G) \geq |V|/\alpha(G)$, and
- $\chi(\overline{G}) \geq \omega(\overline{G})$,

where $\alpha(G)$ is the largest size of an independent set in G, and $\omega(\overline{G})$ is the largest size of a complete subgraph in $\overline{G}$. By definition, $\alpha(G) = \omega(\overline{G})$, hence the two bounds combine into

$$\chi(G) \cdot \chi(\overline{G}) \geq \frac{|V|}{\alpha(G)} \cdot \omega(\overline{G}) = |V|.$$

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2. Let G be a simple graph such that for any two odd cycles C_1 and C_2 in G it holds that $V(C_1) \cap V(C_2) \neq \emptyset$. Prove that $\chi(G) \leq 5$. *(3 points)*

Solution: If G contains no odd cycle, then G is bipartite and $\chi(G) \leq 2$. Otherwise, let C be an odd cycle of G of the shortest length. The subgraph induced by the vertices of C cannot contain any additional edges except the ones from the cycle, as otherwise we would have found a shorter odd cycle. On the other hand, let $W := V(G) \setminus V(C)$. If the induced subgraph $G[W]$ would contain an odd cycle, say C', then we would have found two odd cycles in G such that $V(C) \cap V(C') = \emptyset$ violating the assumption on G. So $G[W]$ is bipartite and can be colored with two colors, say $\{1, 2\}$. The vertices of C can be colored with three new colors, say $\{3, 4, 5\}$. So together this forms a proper 5-coloring of G.

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3. Prove that there exists a constant $C > 0$ such that the following is true: If G is an n-vertex graph with no subgraph isomorphic to K_4, then $\chi(G) \leq C \cdot n^{2/3}$. *(4 points)*

Solution: We prove the statement for $C = 7$, and proceed by induction on n. Firstly, observe that the statement is true for every $n \leq 7^3$. Indeed, for all such n-vertex graphs G we have $7n^{2/3} \geq n \geq \chi(G)$.
Now suppose $n > 7^3$. Without loss of generality, we may assume G is connected, as otherwise we apply induction on its connected components and combine their proper colorings together. Also observe that if G is an odd cycle, then $\chi(G) = 3 \leq 7 \leq 7n^{2/3}$, and if G is complete, then $n \leq 3$. Therefore, if the maximum degree of G is at most $\lfloor 7n^{2/3} \rfloor$, then $\chi(G) \leq 7n^{2/3}$ by Brooks' theorem.
Ok, so it remains to analyze the situation when G is connected, have $n > 7^3$ vertices and contains a vertex v of degree at least $\lfloor 7n^{2/3} \rfloor + 1$. Let N be the set of $\lfloor 7n^{2/3} \rfloor + 1$ arbitrary neighbors of v. Note that $7n^{2/3} \leq |N| \leq 8n^{2/3}$. On the one hand, by the induction hypothesis

$$\chi(G - N) \leq 7(n - |N|)^{2/3} \leq 7\left(n - 7n^{2/3}\right)^{2/3}.$$

On the other hand, the induced subgraph $G[N]$ is triangle-free (as otherwise a triangle in $G[N]$ together with the vertex v creates K_4 in G), hence

$$\chi(G[N]) \leq \sqrt{2|N|} \leq 4n^{1/3}.$$

Therefore, $\chi(G) \leq 7\left(n - 7n^{2/3}\right)^{2/3} + 4n^{1/3}$. We claim that $7\left(n - 7n^{2/3}\right)^{2/3} + 4n^{1/3} \leq 7n^{2/3}$ for all $n > 7^3$. Indeed, we rewrite and use binomial theorem...

$$\begin{aligned} 7\left(n - 7n^{2/3}\right)^{2/3} &\leq 7n^{2/3} - 4n^{1/3} \\ 7^3\left(n - 7n^{2/3}\right)^{2} &\leq \left(7n^{2/3} - 4n^{1/3}\right)^{3} \\ 7^3\left(n^2 - 14n^{5/3} + 49n^{4/3}\right) &\leq 7^3 \cdot n^2 - 7^2 \cdot 12n^{5/3} + 21 \cdot 16n^{4/3} - 64n\ . \end{aligned}$$

Now by rearranging and since $21 \cdot 16n^{4/3} > 64n$, we conclude it is enough to show that

$$7\left(14n^{5/3} - 49n^{4/3}\right) \geq 12n^{5/3}.$$

This clearly holds: since $n^{1/3} > 7$, we divide everything by $n^{5/3}$ and lower-bound the LHS

$$7 \cdot \left(14 - \frac{49}{n^{1/3}}\right) > 49,$$

which clearly beats 12, yay!