MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #10: Proper edge-colorings of graphs

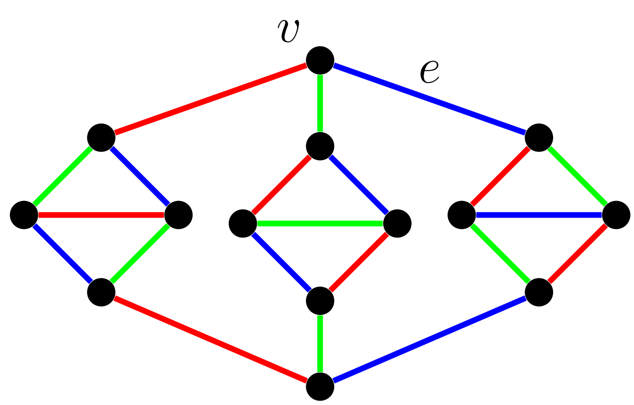
Due Thursday, November 23st, 8:30AM *Write your answers clearly. Justify all your answers.*

1a) Prove that if G is a 3-regular simple graph that contains a Hamilton cycle, then $\chi'(G) = 3$. *(2 points)*

Solution: Let C be a Hamilton cycle in G. By the handshaking lemma every 3-regular graph must have even number of vertices. Therefore, we can properly 2-edge-color the edges of C. Moreover, the subgraph $G - C$ is 1 regular, i.e., it is a perfect matching whose edges we use as the third color class.

1b) Construct a simple 3-regular graph with $\chi'(G) = 3$ that contains no Hamilton cycle. *(1 point)*

Solution: See the 3-regular properly 3-edge-colored graph G depicted here.



(The drawing used and slightly modified with courtesy of Robin Guzniczak.)

Suppose G would have a Hamilton cycle C. Then, in particular, one edge incident with the top vertex v is not be contained in C. Without loss of generality, it is the most-right (blue-colored) edge e. That means that C is a Hamilton cycle also in the graph $G - e$. However, the graph $G - e$ is not 2-edge-connected and therefore has no Hamilton cycle; a contradiction.

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2. For $n \geq 2$, use the following steps to determine $\chi'(K_n)$ and construct its optimal edge-coloring:

a) For every <u>odd</u> integer $n \geq 3$, observe that K_n does not have an edge-coloring with $n-1$ colors. *(1 point)*

Solution: Indeed, K_n is an $(n-1)$-regular graph, so if it has an edge-coloring with $n-1$ colors, then each color class must form a perfect matching. But for n odd, K_n cannot have a perfect matching.

b) For every <u>odd</u> integer $n \geq 3$, prove that if c is an edge-coloring of K_n with n colors, then each color class of c contains $(n-1)/2$ edges. (Note that $\chi'(K_n) = n$ follows from Vizing's Theorem) *(1 point)*

Solution: Consider an edge-coloring of K_n with n colors. Each color class is a matching, and since n is odd, any matching of K_n has size at most $(n-1)/2$ edges. However, each edge of K_n has one of the n colors and since

$$\binom{n}{2} = n \cdot \frac{n-1}{2},$$

we conclude that the bound $(n-1)/2$ on the size of a color class must be tight.

c) For every <u>even</u> integer $n \geq 2$, use (b) to show that $\chi'(K_n) = n - 1$. *(1 point)*

Solution: Consider any edge-coloring of K_{n-1} using $n-1$ colors. From the part (b), we know that each color class contains $(n-2)/2$ edges. In other words, for each color $i \in \{1, \ldots, n-1\}$, there is exactly one vertex v_i that is not incident to any edge colored with i. Moreover, for different colors $i \neq j$, it holds that $v_i \neq v_j$. Adding a new vertex v_n and coloring the edge $\{v_i, v_n\}$ with the color i for all $i \in \{1, \ldots, n-1\}$ yields an $(n-1)$-edge-coloring of K_n.

d) For every integer $n \geq 2$, explicitly construct an edge-coloring of K_n with $\chi'(K_n)$ colors. *(1 point)*

[*Hint for (d): if n is odd, put $V(K_n) = \{0, \ldots, n-1\}$ and color the edge $\{i, j\}$ with $(i + j) \mod n$.*]

Solution: As the hint suggested, we should show that for n being odd and $V(K_n) = \{0, \ldots, n-1\}$, coloring the edge $\{i, j\}$ with $(i + j) \mod n$ yields an edge-coloring of K_n. Suppose for a contradiction that there are two edges $e_1 \neq e_2$ incident to some vertex i that are both colored with the same color, say $x \in \{0, \ldots, n-1\}$. Let $e_1 = \{i, j\}$ and $e_2 = \{i, k\}$. Since $(i + j) \equiv x \equiv (i + k) \mod n$, we have $j \equiv k \mod n$. However, that means that $j = k$ contradicting $e_1 \neq e_2$.

If n is even, we let $n' := n - 1$ and $V(K_n) = \{0, \ldots, n'-1, n'\}$. If $i, j \in \{0, \ldots, n'-1\}$, we color the edge $\{i, j\}$ with $(i + j) \mod n'$, and the remaining edges $\{i, n'\}$, where $i \in \{0, \ldots, n'-1\}$, we color with $(2i) \mod n'$. Since n' is odd, it follows that $2i \neq 2j \mod n'$ for any $i, j \in \{0, \ldots, n'-1\}$ with $i \neq j$.

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3. Let $G = (V, E)$ be a loopless multigraph. Recall the *line graph* of G, which we denote by $L(G)$, is a simple graph with the vertex set being E, and $e \in E$ is adjacent to $f \in E$ in $L(G)$ if and only if the edges e and f of G have an endpoint in common. Equivalently, $L(G) = (E, F)$ where $F = \{\{e, f\} : e \cap f \neq \emptyset\}$.

a) Let $G = (V, E)$ be a loopless connected multigraph with an <u>even</u> number of edges. Prove that the line graph $L(G)$ has a perfect matching. *(2 points)*

[*Hint for (a): use Tutte's Theorem.*]

Solution: Suppose for contradiction $L(G)$ does not have a perfect matching. By Tutte's theorem, there exists $S \subseteq E$ such that $k > |S|$ for $k := odd_{L(G)}(E \setminus S)$. It follows that the parity of k is the same as the parity of $|S|$, hence $k \geq |S| + 2$. Now look back to the graph G. The connected components of the subgraph of $L(G)$ induced by $E \setminus S$ are in one-to-one correspondence with the connected components of $G' := (V, E \setminus S)$. So G' has at least k connected components. However, each edge from S can connect at most two components of G' and since $|S| < k - 1$, G cannot be connected.

b) Let $G = (V, E)$ be a loopless connected multigraph with an <u>odd</u> number of edges. Prove that $L(G)$ has a matching of size $\frac{|E|-1}{2}$. *(1 point)*

Solution: Simply add an aribtrary edge to G connecting two different vertices and use the previous part. The perfect matching M in the line graph of the new graph contains a matching $M' \subseteq E$ of size $\frac{|E|-1}{2}$.

Alternatively, if G is not a tree, there is $e \in E$ such that $G' := G - e$ is connected. On the other hand, if G is a tree, then let v be a leaf and $G' := G - v$. In both cases, G' is connected $|E(G')|$ is even, and $L(G')$ is a subgraph of $L(G)$, so we use the part a).