1a) Prove that if $G$ is a 3-regular simple graph that contains a Hamilton cycle, then $\chi'(G) = 3$. (2 points)

Solution: Let $C$ be a Hamilton cycle in $G$. By the handshaking lemma every 3-regular graph must have even number of vertices. Therefore, we can properly 2-edge-color the edges of $C$. Moreover, the subgraph $G - C$ is 1 regular, i.e., it is a perfect matching whose edges we use as the third color class.

1b) Construct a simple 3-regular graph with $\chi'(G) = 3$ that contains no Hamilton cycle. (1 point)

Solution: See the 3-regular properly 3-edge-colored graph $G$ depicted here.

![Graph Image]

(The drawing used and slightly modified with courtesy of Robin Guzniczak.)

Suppose $G$ would have a Hamilton cycle $C$. Then, in particular, one edge incident with the top vertex $v$ is not be contained in $C$. Without loss of generality, it is the most-right (blue-colored) edge $e$. That means that $C$ is a Hamilton cycle also in the graph $G - e$. However, the graph $G - e$ is not 2-edge-connected and therefore has no Hamilton cycle; a contradiction.

2. For $n \geq 2$, use the following steps to determine $\chi'(K_n)$ and construct its optimal edge-coloring:

a) For every odd integer $n \geq 3$, observe that $K_n$ does not have an edge-coloring with $n - 1$ colors. (1 point)

Solution: Indeed, $K_n$ is an $(n - 1)$-regular graph, so if it has an edge-coloring with $n - 1$ colors, then each color class must form a perfect matching. But for $n$ odd, $K_n$ cannot have a perfect matching.

b) For every odd integer $n \geq 3$, prove that if $c$ is an edge-coloring of $K_n$ with $n$ colors, then each color class of $c$ contains $(n - 1)/2$ edges. (Note that $\chi'(K_n) = n$ follows from Vizing’s Theorem) (1 point)

Solution: Consider an edge-coloring of $K_n$ with $n$ colors. Each color class is a matching, and since $n$ is odd, any matching of $K_n$ has size at most $(n - 1)/2$ edges. However, each edge of $K_n$ has one of the $n$ colors and since

$$\binom{n}{2} = n \cdot \frac{n - 1}{2},$$

we conclude that the bound $(n - 1)/2$ on the size of a color class must be tight.
c) For every even integer \( n \geq 2 \), use (b) to show that \( \chi'(K_n) = n - 1 \).  

**Solution:** Consider any edge-coloring of \( K_{n-1} \) using \( n - 1 \) colors. From the part (b), we know that each color class contains \( (n - 2)/2 \) edges. In other words, for each color \( i \in \{1, \ldots, n-1\} \), there is exactly one vertex \( v_i \) that is not incident to any edge colored with \( i \). Moreover, for different colors \( i \neq j \), it holds that \( v_i \neq v_j \). Adding a new vertex \( v_n \) and coloring the edge \( \{v_i, v_n\} \) with the color \( i \) for all \( i \in \{1, \ldots, n-1\} \) yields an \( (n - 1) \)-edge-coloring of \( K_n \).

\[ \chi'(K_n) = n - 1. \]

(1 point)

\[ \text{Hint for (a): use Tutte’s Theorem.} \]

\[ \text{Hint for (d): if } n \text{ is odd, put } V(K_n) = \{0, \ldots, n-1\} \text{ and color the edge } \{i, j\} \text{ with } (i + j) \mod n. \]

**Solution:** As the hint suggested, we should show that for \( n \) being odd and \( V(K_n) = \{0, \ldots, n-1\} \), coloring the edge \( \{i, j\} \) with \( (i + j) \mod n \) yields an edge-coloring of \( K_n \). Suppose for a contradiction that there are two edges \( e_1 \neq e_2 \) incident to some vertex \( i \) that are both colored with the same color, say \( x \in \{0, \ldots, n-1\} \). Let \( e_1 = \{i, j\} \) and \( e_2 = \{i, k\} \). Since \( (i + j) \equiv x \equiv (i + k) \mod n \), we have \( j \equiv k \mod n \). However, that means that \( j = k \) contradicting \( e_1 \neq e_2 \).

If \( n \) is even, we let \( n' := n - 1 \) and \( V(K_n) = \{0, \ldots, n' - 1\} \). If \( i, j \in \{0, \ldots, n' - 1\} \), we color the edge \( \{i, j\} \) with \( (i + j) \mod n' \), and the remaining edges \( \{i, n'\} \), where \( i \in \{0, \ldots, n' - 1\} \), we color with \( (2i) \mod n' \). Since \( n' \) is odd, it follows that \( 2i \neq 2j \mod n' \) for any \( i, j \in \{0, \ldots, n' - 1\} \) with \( i \neq j \).

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3. Let \( G = (V, E) \) be a loopless multigraph. Recall the line graph of \( G \), which we denote by \( L(G) \), is a simple graph with the vertex set being \( E \), and \( e \in E \) is adjacent to \( f \in E \) in \( L(G) \) if and only if the edges \( e \) and \( f \) of \( G \) have an endpoint in common. Equivalently, \( L(G) = (E, F) \) where \( F = \{\{e, f\} : e \cap f \neq \emptyset\} \).

a) Let \( G = (V, E) \) be a loopless connected multigraph with an even number of edges. Prove that the line graph \( L(G) \) has a perfect matching.  

**Solution:** Suppose for contradiction \( L(G) \) does not have a perfect matching. By Tutte’s theorem, there exists \( S \subseteq E \) such that \( k > |S| \) for \( k := \text{odd}_{L(G)}(E \setminus S) \). It follows that the parity of \( k \) is the same as the parity of \( |S| \), hence \( k \geq |S| + 2 \). Now look back to the graph \( G \). The connected components of the subgraph of \( L(G) \) induced by \( E \setminus S \) are in one-to-one correspondence with the connected components of \( G' := (V, E \setminus S) \). So \( G' \) has at least \( k \) connected components. However, each edge from \( S \) can connect at most two components of \( G' \) and since \( |S| < k - 1 \), \( G \) cannot be connected.

b) Let \( G = (V, E) \) be a loopless connected multigraph with an odd number of edges. Prove that \( L(G) \) has a matching of size \( \frac{|E| - 1}{2} \).  

**Solution:** Simply add an arbitrary edge to \( G \) connecting two different vertices and use the previous part. The perfect matching \( M \) in the line graph of the new graph contains a matching \( M' \subseteq E \) of size \( \frac{|E| - 1}{2} \).

Alternatively, if \( G \) is not a tree, there is \( e \in E \) such that \( G' := G - e \) is connected. On the other hand, if \( G \) is a tree, then let \( v \) be a leaf and \( G' := G - v \). In both cases, \( G' \) is connected \( |E(G')| \) is even, and \( L(G') \) is a subgraph of \( L(G) \), so we use the part a).