MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #11: Planar graphs

Due Thursday, November 30st, 8:30AM *Write your answers clearly. Justify all your answers.*

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1. Let G be a simple triangle-free planar graph. Without using the 4-Color Theorem, prove that $\chi(G) \leq 4$. *(2 points)*

Solution: We prove the statement by induction on $n := |V(G)|$. If $n \leq 4$, then the statement is indeed vacuously true. In the rest, we assume $n \geq 5$. By the triangle-free part of Lemma 17.4 in the lecture notes, we know that G has at most $2n - 4$ edges. Therefore, G has a vertex v of degree at most 3, as otherwise

$$4n - 8 \geq 2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq 4n\,;$$

a contradiction. So by using the induction hypothesis on the subgraph $G' := G - v$ (which is clearly triangle-free), we obtain a proper 4-coloring c' of G', which we simply extend to G by coloring the vertex v with one of the colors not used on its neighbors in c'. (Alternatively, we could have used the above reasoning to prove that G is 3-degenerate.)

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2. A simple graph G is called *outerplanar* if it can be drawn in the plane without any crossing in such a way that every vertex is incident with the infinite region.

Let $G = (V, E)$ be a connected outerplanar graph with $|V| \geq 3$.

a) Prove that G contains two vertices of degree at most 2. *(1 point)*

Solution: If G has 3 vertices, then every vertex has the degree at most 2. For an outerplanar G on at least 4 vertices, we prove the following stronger lemma:

Lemma. *If $|V(G)| \geq 4$, then G contains at least two non-adjacent vertices of degree at most 2.*

We proceed by induction on n. If the number of vertices is equal to 4, then the only way how G could avoid having a pair of non-adjacent vertices of degree at most 2 is when $G = K_4$. However, K_4 cannot be outerplanar: take its arbitrary outerplanar drawing, place a new vertex w in the infinite region, and connect w with all the vertices of K_4. This yields a plane drawing of K_5; a contradiction.

Now suppose $|V| \geq 5$. If G is disconnected, then G has at least two connected components and each component of G contains at least one vertex of degree at most 2 (if the number of the vertices in the component is at most 3, then the statement follows trivially, otherwise we use the induction hypothesis). If G contains a cut-vertex v, then let C_1 be one of the connected components of $G - v$, $V_1 := V(C_1) \cup \{v\}$ and $V_2 := V \setminus V(C_1)$. Let G_1 and G_2 be the subgraph of G induced by V_1 and V_2, respectively. We claim that both G_1 and G_2 contains a vertex $v_1 \neq v$ and $v_2 \neq v$, respectively, of degree at most 2 (clearly, v_i has the same degree in G_i and G for $i \in \{1, 2\}$). Indeed, for both $i = 1$ and $i = 2$, if $|V_i| \leq 3$, then every vertex of G_i has degree at most 2, so just select an arbitrary $v_i \in V_i \setminus \{v\}$. On the other hand, if $|V_i| \geq 4$, then by the induction hypothesis G_i contains two vertices of degree at most 2, so let v_i be one of the vertices that is not v.

It remains to analyze the case that G is 2-connected. Firstly, we observe that any 2-connected outerplanar graphs contains a Hamiltonian cycle. Indeed, take any of outerplanar drawing of G, and let C be a longest cycle in G. If $V(C) = V(G)$, then we are done. Otherwise let $x \in V(G) \setminus V(C)$ and let P_1 and P_2 be two paths from x to $v_1, v_2 \in V(C)$, respectively, such that $V(P_1) \cap V(P_2) = \{x\}$. Note that these two paths are guaranteed to exist by the 2-connectivity of G. But then either v_1 and v_2 are consecutive on C, which means G contains a cycle longer than G, or, take any two vertices $x, y \in V(C)$ such that the cyclic order of vertices on C is $v_1, \ldots, y, \ldots, v_2, \ldots, z, \ldots, v_1$. It follows that v_1, v_2, x, y, z is a subdivision of $K_{2,3}$, and hence any outerplanar drawing of $G[C \cup \{x\}]$ extended by placing a new vertex w in the infinite region and connecting it to x, y, z yields a planar drawing of a subdivision of $K_{3,3}$; a contradiction.

Now, let C be a Hamilton cycle in G. If $E(G) = E(C)$, then G is 2-regular and since $|V| \geq 5$, we can select any two non-adjacent vertices of G. Otherwise, let u and w be two adjacent vertices in G such that $\{u, w\} \notin E(C)$. Observe that in any outerplanar drawing of G, all the additional edges must be drawn "inside" C. Therefore, $G - \{u, w\}$ is disconnected, and let C_1 be one of the corresponding connected components. Analogously to the previous case, we set $V_1 := V(C_1) \cup \{u, w\}$ and $V_2 := V \setminus V(C_1)$, and let $G_1 := G[V_1]$ and $G_2 := G[V_2]$. Now we claim for both $i = 1$ and $i = 2$, G_i contains a vertex v_i of degree at most 2 that is neither u nor w. Indeed, if $|V_i| = 3$, then choose the third vertex in V_i to be v_i. If $|V_i| \geq 4$, then G_i contains at least two non-adjacent vertices x, y with degrees at most 2. Since u and w are adjacent, it follows that $\{x, y\} \neq \{u, w\}$. Choose $v_i \in \{x, y\} \setminus \{u, w\}$ arbitrarily; in all the cases, v_1 and v_2 are non-adjacent and have degrees at most 2.

b) Without using the 4-Color Theorem, prove that $\chi(G) \leq 3$. *(1 points)*

Solution. The part (a) yields that G is 2-degenerate so indeed, $\chi(G) \leq 3$.

c) Prove that a graph is outerplanar if and only if it contains no K_4-minor and no $K_{2,3}$-minor. *(2 points)*

Solution. Consider the graph G^+ which is obtained from G by adding a new vertex v and connecting v to all the vertices of G. First of all, if G is outerplanar, we claim that G^+ is planar. Indeed, consider an outerplanar drawing of G in the plane, draw v in the infinite region, and simply connect v to all the vertices of G so that the edges do not cross. We have found a plane drawing of G^+ so it is planar. However, if G would contain either a minor of K_4 or a minor $K_{2,3}$, we can easily find a minor of K_5 or $K_{3,3}$ in G^+, contradicting Kuratowski's theorem.

Now we essentially flip this argument in order to show the other implication. If G does not contain a minor of K_4 or $K_{2,3}$, then G^+ contains neither a minor of K_5 nor $K_{3,3}$. So by Kuratowski's theorem, G^+ is planar. If a drawing of G^+ is such that v is not on the boundary of the infinite region, then consider any region R with v on its boundary and apply so-called circle inversion to obtain a new drawing of G^+ in the plane so that everything that was drawn outside of R is now inside. In this way, we obtained a drawing D of G^+ with v on the boundary of the infinite region, and it immediately follows that if delete v and all of the edges incident to v from D, we obtain an outerplanar drawing of G. So in particular, G is outerplanar.

3a) Let H be a simple graph with maximum degree at most 3. Show that every simple graph contains a subdivision of H if and only if it contains H as a minor. *(2 points)*

3b) Let G be a simple graph that contains K_5 as a minor. Prove that G contains a subdivision of K_5 or a subdivision of $K_{3,3}$. *(2 points)*

Solution: Before proving the parts 3a and 3b, it will be useful to have the following lemma

Lemma. *Let $H = (W, F)$ with $W = \{w_1, w_2, \ldots, w_k\}$ be contained in $G = (V, E)$ as a minor. Then there exists vertex-disjoint sets $W_1, W_2, \ldots, W_k \subseteq V$ such that:*

1. *The induced subgraph $G[V_i]$ is connected, and*
2. *if $\{w_i, w_j\} \in F$, then there exist $v_i \in W_i$ and $v_j \in W_j$ such that $e_{ij} = \{v_i, v_j\} \in E$.*

Proof. Without loss of generality, we may assume H was obtained from G only by a sequence of contracting edges as otherwise we would find the desired vertices and edges in the appropriate subgraph of G. Let $e_1, e_2, \ldots, e_\ell$ be the sequence of edge-contractions such that $H_0 := G$, H_i is obtained from H_{i-1} by contracting $e_i \in E(H_{i-1})$ for all $i \in \{1, 2, \ldots, \ell\}$, and $H_\ell = H$. We prove the lemma by induction on ℓ. Clearly, if $\ell = 0$, then $H = G$ and the lemma trivially holds. Otherwise let $e_1 = \{u, w\} \in E(G)$ and recall H_1 was obtained from G by contracting e_1.

Applying the induction hypothesis on the graph H_1, we get k disjoint sets $W'_1, W'_2, \ldots, W'_k \subseteq V(H_1)$ with the desired property. Let $x \in V(H_1)$ be the vertex created from contracting $\{u, w\}$. If $x \notin \bigcup W'_i$, then the W'_i, for $i \in \{1, 2, \ldots, k\}$ are the sought sets also for G. Assume, without loss of generality, $x \in W'_1$. But then let $W_1 := (W'_1 \setminus \{x\}) \cup \{u, w\}$ and $W_i := W'_i$ for all $i \geq 2$. It is straightforward to check the sets W_i have all the desired properties. □

Armed with this lemma, we proceed firstly with proving the part 3a. Fro any H, If G contains a subdivision of H, then it clearly contains H as a minor, so we move to the other implication (where we will of course use the maximum-degree bound). Let $H = (W, F)$ with $W = \{w_1, w_2, \ldots, w_k\}$ be contained in G as a minor, and let $W_1, W_2, \ldots, W_k$ be the disjoint sets of vertices of G from the lemma, and $e_{ij} \in E(G)$, for all i, j such that $\{w_i, w_j\} \in E(H)$, the edges from the lemma. For each set W_i, we find a vertex $z_i \in W_i$, and for each edge $\{w_i, w_j\} \in E(H)$, we find a path P_{ij} in G connecting z_i to z_j that is internally vertex-disjoint from the paths $P_{i'j'}$ for $i'j' \neq ij$. This forms a subdivision of H. We remark that each path P_{ij} will be actually consisting of three pieces: a path $P_i^j \subseteq W_i$, which connects z_i to the unique vertex in $e_{ij} \cap W_i$, the edge e_{ij} itself, and a path $P_j^i \subseteq W_j$ connecting $e_{ij} \cap W_j$ to z_j.
For any $i \in \{1, 2, \ldots, k\}$ we have that w_i has degree at most 3, hence there are at most 3 edges in G from the second part of the outcome of the lemma that have one of the endpoints in W_i. If there is only one such edge, say e_{ij}, i.e., w_i has degree 1, then simply take the endpoint of e_{ij} inside W_i to be z_i, and $P_i^j := \emptyset$. If the degree of w_i in H is 2, then let w_j and w_k be its neighbors in H, $x_j, x_k \in W_i$ the endpoints in W_i of the edges e_{ij} and e_{ik}, respectively, given by the lemma. Note that it can be $x_j = x_k$. We set $z_i := x_j$, $P_i^j := \emptyset$, and P_i^k to be a shortest path inside W_i from x_j to x_k. Finally, if w_i has degree 3 in H, then let w_j, w_k, and w_ℓ be its neighbors, and $x_j, x_k, x_\ell \in W_i$ the endpoints of the edges from the lemma. Let Q be a shortest path inside W_i that connects x_j and x_k, Q' a shortest path from x_ℓ to a vertex of $V(Q)$, and let z_i be the unique vertex that is both in Q and Q'. We define P_i^j and P_i^k to be the subpaths of Q from z_i to x_j and x_k, respectively, and finally $P_i^\ell := Q'$. It readily follows that we just found a subdivision of H.

The part 3b could be either deduced from the combination of the part 3a, Kuratowski's and Kuratowski-Wagner's theorems, or, if we want an independent proof (which is needed if our aim is actually to prove the equivalence of the two theorems), we proceed as follows: Apply the lemma for G and $H := K_5$ to get 5 vertex-disjoint sets $W_1, W_2, \ldots, W_5 \subseteq V(G)$ that are connected, and the edges $e_{ij} \in E(G)$ for $1 \leq i < j \leq 5$ between them as described in the statement of the lemma. Fix $i \in \{1, 2, 3, 4, 5\}$ arbitrarily, and let x_i^j, for $1 \leq j \leq 5$ and $j \neq i$, be the vertices that are the endpoints of the edges e_{ij} inside W_i. Suppose first that for each i, the induced subgraph $G[W_i]$ contains a vertex z_i and four internally vertex-disjoint paths P_i^j to all the four vertices x_i^j. In this case, we can combine these paths together with the edges e_{ij} to get a subdivision of K_5 in the same way as we did for H in the part 3a.
For the rest of the proof, suppose there is $i \in \{1, 2, 3, 4, 5\}$ such that W_i has no such a vertex z_i. Without loss of generality, $i = 1$. Let $x^2, x^3, x^4, x^5 \in W_1$ be the four vertices incident to the edges $e_{12}, e_{13}, e_{14}, e_{15}$, respectively. Now, let Q be a shortest path in $G[W_1]$ from x^2 to x^3, and Q' a shortest path from x^4 to some vertex of Q. Let z_a be the unique vertex of $V(Q) \cap V(Q')$, and P^2, P^3, P^4 be the unique paths from z_a to x_2, x_3, x_4, respectively, that lie inside $V(Q) \cup V(Q')$. Now let Q'' be a shortest path from x^5 some vertex in $V(Q) \cup V(Q')$, and let z_b be the endpoint of Q'' different from x^5. It follows that $z_a \neq z_b$ as otherwise we have just found a vertex z_1 with the property of having four vertex-disjoint paths to all $x^2, \ldots, x^5$. Without loss of generality, z_b lies somewhere on P^4. Let R be the subpath of P^4 between z_a and z_b, and $f \in E(G)$ an arbitrary edge on R, for example the edge from z_a. Let's consider an arbitrary spanning tree T_1 of W_1 that contains the edge f, and spanning trees T_2, T_3, T_4, T_5 of W_2, W_3, W_4, W_5, respectively, and let H be a minor of G obtained by contracting $\bigcup_{i=1}^{5} E(T_i) \setminus f$.
Note that H is a 6 vertex graph, and we denote $V(H) = \{z_a, z_b, w_2, w_3, w_4, w_5\}$. It follows that the vertices w_2, w_3, w_4, w_5 induces K_4 in H, $\{z_b, w_2, w_3\} \subseteq N(z_a)$, and $\{z_a, w_4, w_5\} \subseteq N(z_b)$. Clearly, H contains a $K_{3,3}$ subgraph with parts $A = \{z_a, w_4, w_5\}$ and $B = \{z_b, w_2, w_3\}$, and hence G contains $K_{3,3}$ as a minor. Therefore, by the part 3a, G also contains $K_{3,3}$ as a subdivision, which is what we needed to prove.