

Forcing quasirandomness in permutations

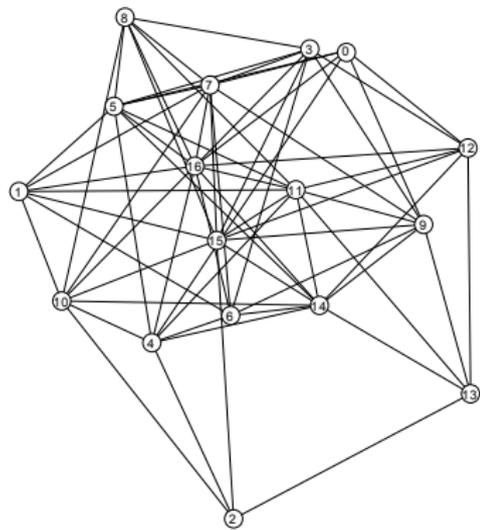
Jan Volec

MSCA fellow at Emory University & Universität Hamburg

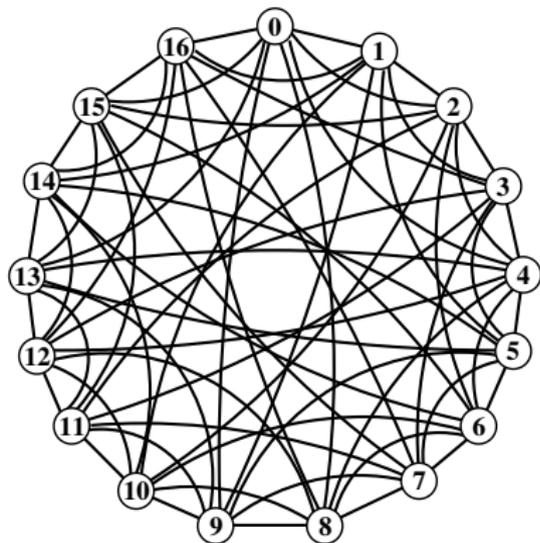
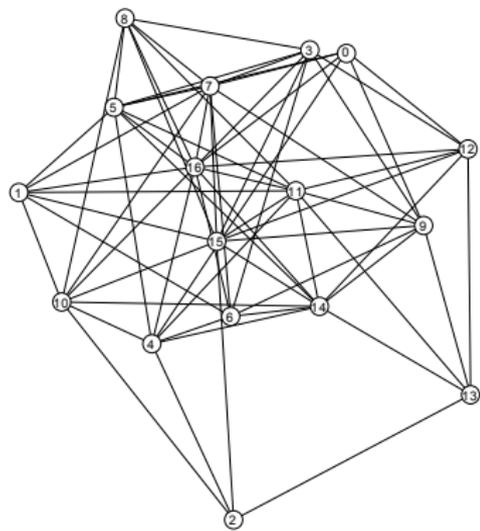
Based on a joint work with T. Chan, D. Král' and J. Noel.

 This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 800607.

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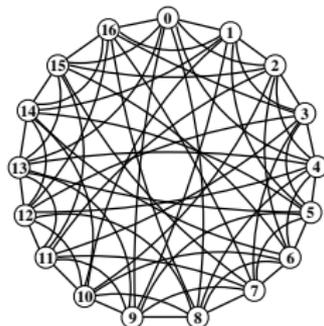
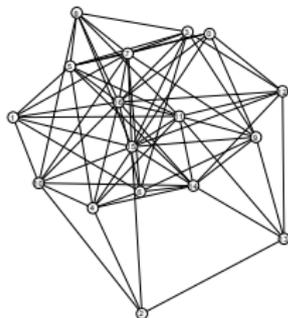


Quasi-random / pseudorandom graphs

Late 80's: Chung-Graham-Wilson extending Rödl and Thomason:

Large n -vertex graph $G = (V, E)$ of density p is quasi-random if:

- $\forall X \subseteq V$, $G[X]$ has $\approx p \cdot \binom{|X|}{2}$ edges
- $\forall A, B \subseteq V$, # of edges between A and B is $\approx p \cdot |A||B|$
- $\forall F$, # of labelled non-induced F in G is $\approx p^{e(F)} \cdot n^{v(F)}$
- For $F = C_4$, # of labelled non-induced $F \subset G$ is $\approx p^4 \cdot n^4$
- $\sum_{u, w \in V} |\text{codeg}(u, w) - p^2 \cdot n| = o(n^3)$
- $\lambda_2(G) = o(n)$

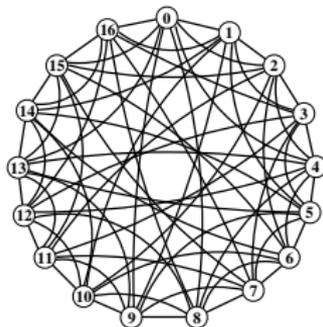
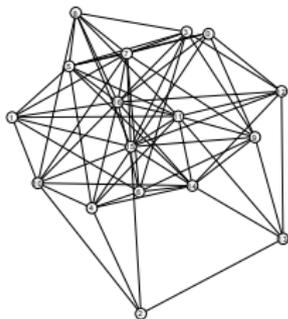


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Quasi-randomness of G with density p by counting C_4 s

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Smallest open case for both conjectures: $B = K_{5,5} - C_{10}$

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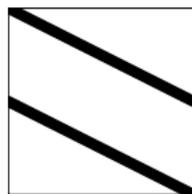
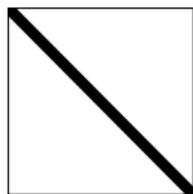
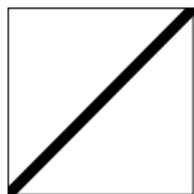
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Therefore, $F(a, b) = ab \cdot c$ for some $c \in \mathbb{R}$. $F(1, 1) = 1 \Rightarrow c = 1$

Reducing the number of assumptions in Král'-Pikhurko

Original proof uses density $d(\tau, \mu)$ of all 24 permutations $\tau \in S_4$

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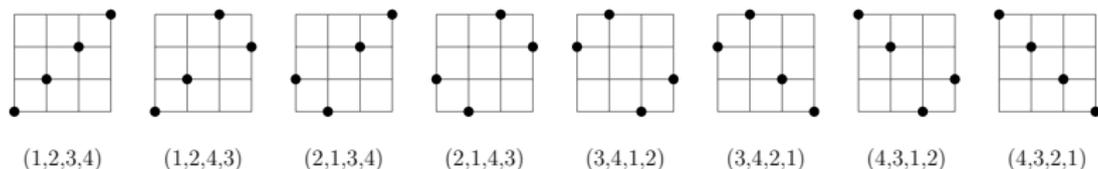
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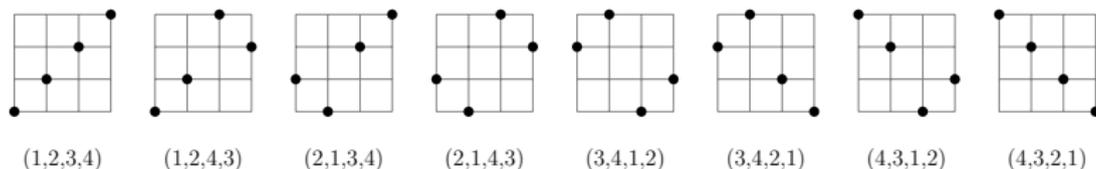
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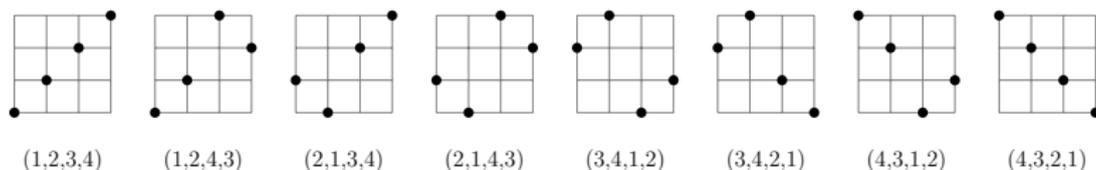
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Each \int express some pattern density. For some $q, r, s > 0$ it holds:

$$3 \cdot \left(\sum_{\tau \in T} d(\tau, \mu) \right) - 1 = q \cdot Q + r \cdot R + s \cdot S$$

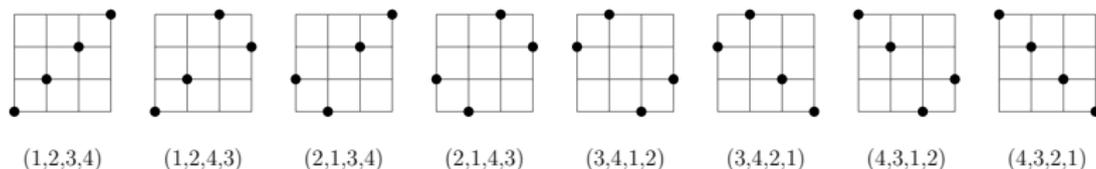
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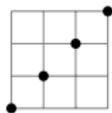
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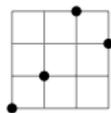
Conclusion

Chan-Král'-Noel-JV: $\sum_{\tau \in \mathcal{T}} d(\tau, \mu) \geq \frac{1}{3} \forall \mu$ on $[0, 1]^2$ w.unif.margin.

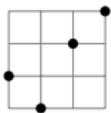
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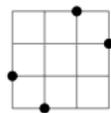
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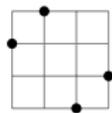
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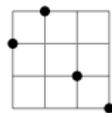
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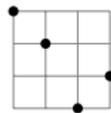
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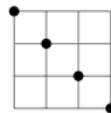
(3,4,1,2)



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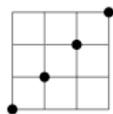


(4,3,2,1)

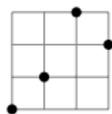
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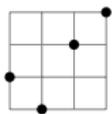
and the \sum is equal to $1/3$ only if μ is the uniform measure



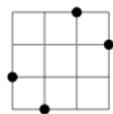
(1,2,3,4)



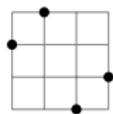
(1,2,4,3)



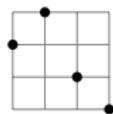
(2,1,3,4)



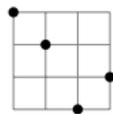
(2,1,4,3)



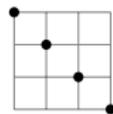
(3,4,1,2)



(3,4,2,1)



(4,3,1,2)



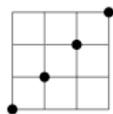
(4,3,2,1)

Question: Is \mathcal{T} minimal with respect to forcing quasi-randomness?

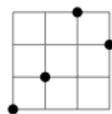
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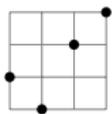
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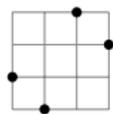
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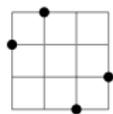
(1,2,4,3)



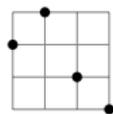
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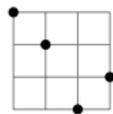
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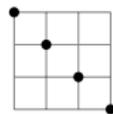
(3,4,1,2)



(3,4,2,1)



(4,3,1,2)



(4,3,2,1)

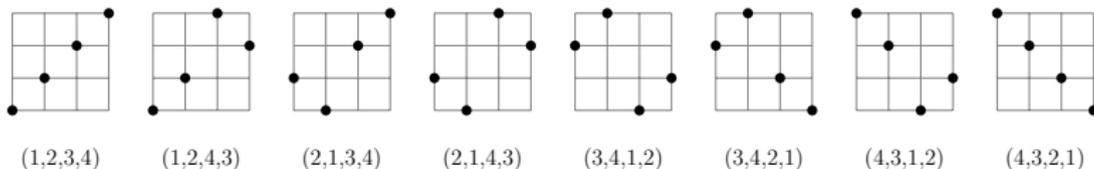
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Chan-Král'-Noel-JV: $\sum_{\tau \in T} d(\tau, \mu) \geq \frac{1}{3} \forall \mu$ on $[0, 1]^2$ w.unif.margin.

and the \sum is equal to $1/3$ only if μ is the uniform measure



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Sidorenko (and forcing) conjecture(s) for B k -vertex bipartite:

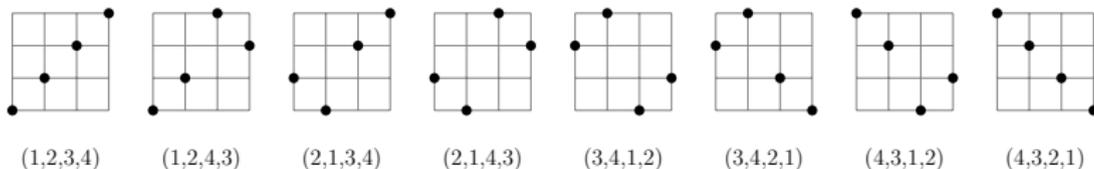
$$\# \text{ of } B \text{ in } n\text{-vertex } G \text{ of density } p \geq p^{e(B)} \cdot n^k$$

Smallest open case $B = K_{5,5} - C_{10}$

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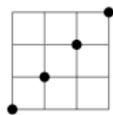
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Question: Is there some permutation-analogue of Sidorenko?

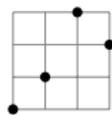
Conclusion Thank you for your attention!

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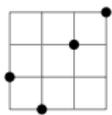
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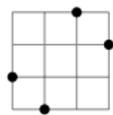
(1,2,3,4)



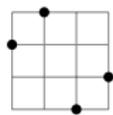
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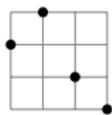
(2,1,3,4)



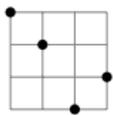
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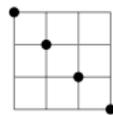
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