On degree thresholds of cycles in oriented graphs

Jan Volec

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joint work with Roman Glebov and Andrzej Grzesik

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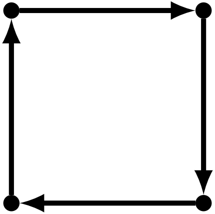

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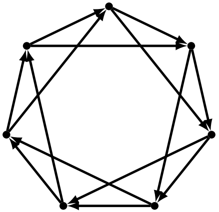

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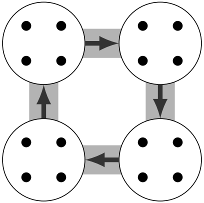

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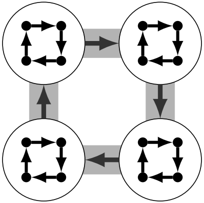

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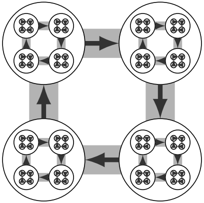

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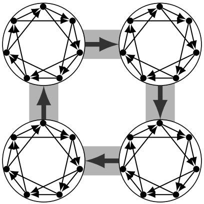

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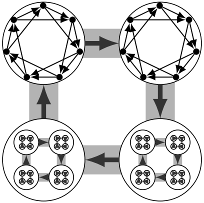

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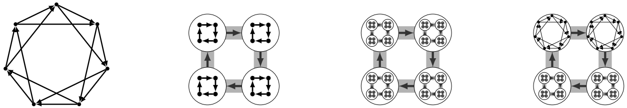

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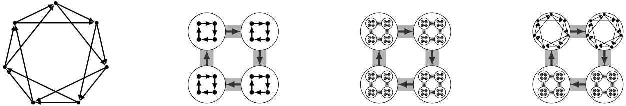

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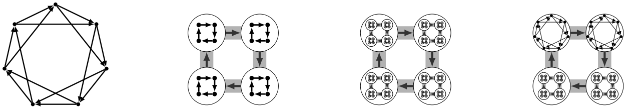

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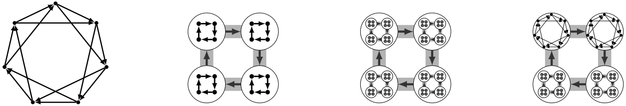

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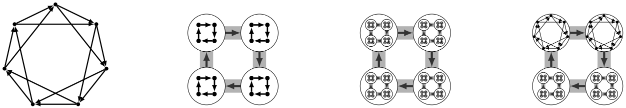

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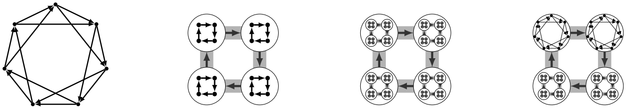

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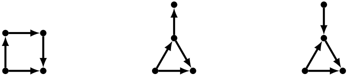

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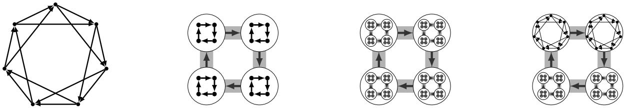

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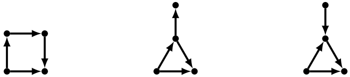

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- Sereni, De Joannis De Verclos, JV: $c < 0.339$

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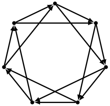
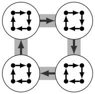
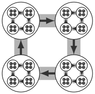
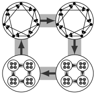

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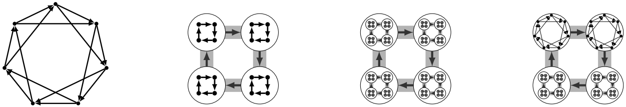

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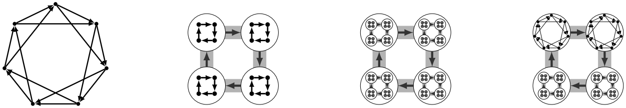

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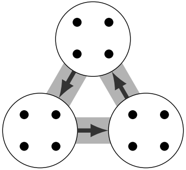

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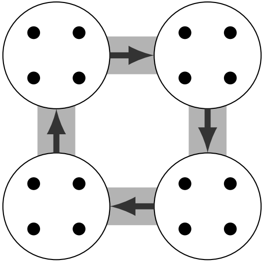

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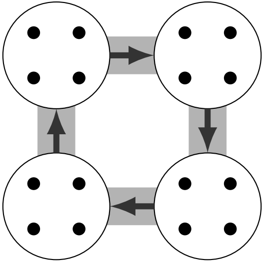

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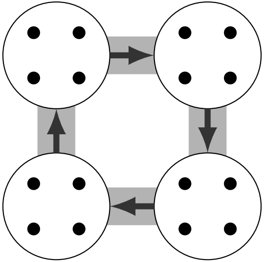

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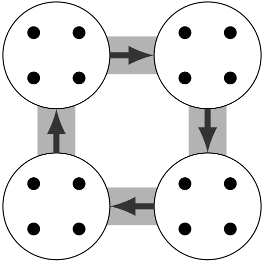

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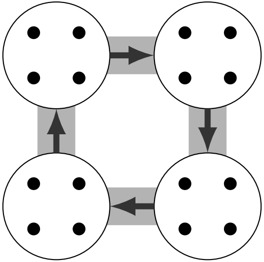

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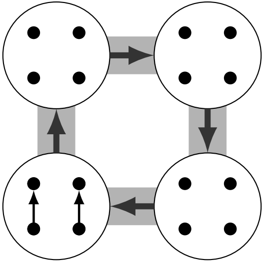

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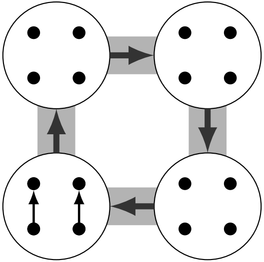

Stronger conjecture: G n-vertex oriented $\delta^{+}(G) \gg \frac{n}{k_\ell} \Rightarrow \vec{C}_\ell \in G$?

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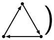
(By  removal lemma, we may also assume no )

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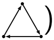
(By  removal lemma, we may also assume no )

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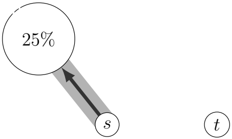

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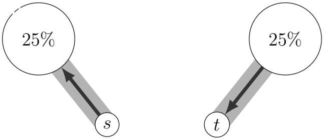

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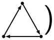
(By  removal lemma, we may also assume no )

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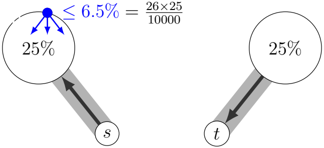

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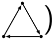
(By  removal lemma, we may also assume no )

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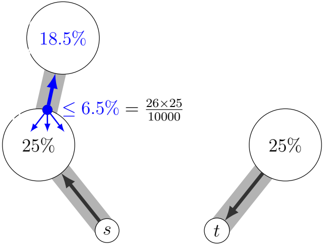

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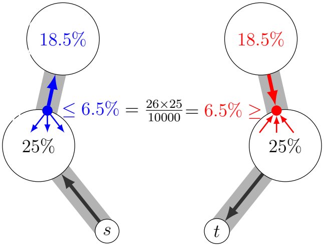

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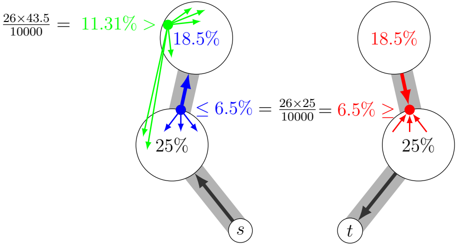

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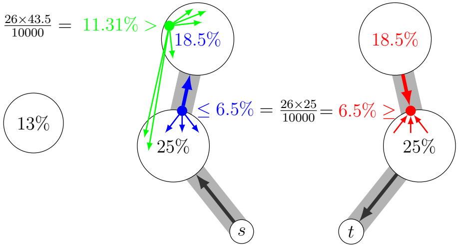

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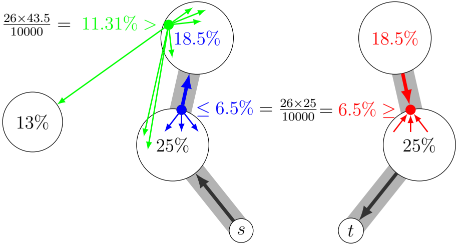

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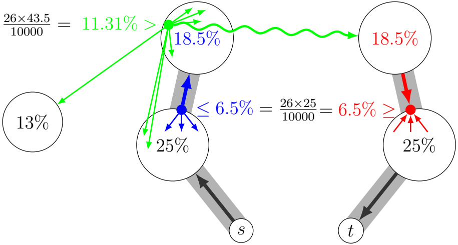

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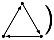
(By  removal lemma, we may also assume no )

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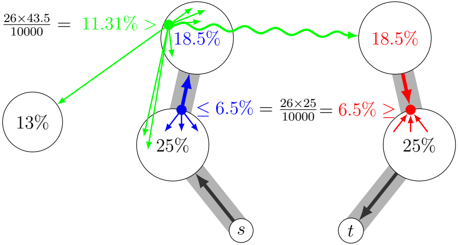

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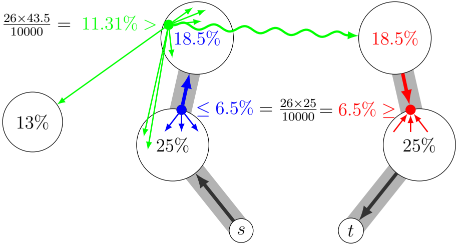

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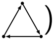
(By  removal lemma, we may also assume no )

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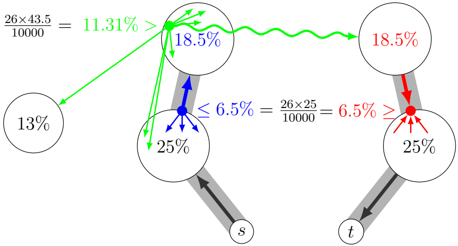

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Corollary'': $\forall s, t$: there is oriented path from s to t of length ≤ 4

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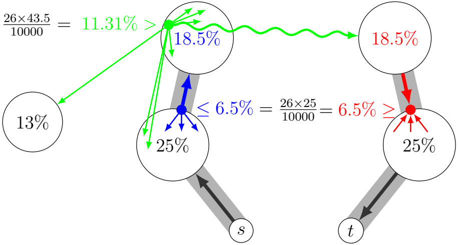

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Corollary": $\forall s, t$: there is oriented path from s to t of length ≤ 4
Do Corollary again but inside $N^{+}(s)$ & $N^{-}(t)$ pick sink (source)

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

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Analyze (dense) large graphs via relations between small subgraphs

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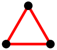
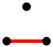
$$\sum_{u} \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#$$  $$+ \#$$  $$\approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

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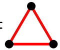
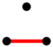
$$0 \leq \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#$$  $$+ \#$$  $$\approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

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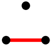
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\triangle + \#$$  $$\approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

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Powerful tool, shed light to many open problems

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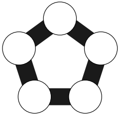

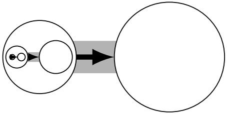

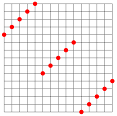

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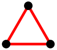
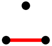
$$0 \leq \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#$$  $$+ \#$$  $$\approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

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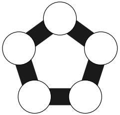

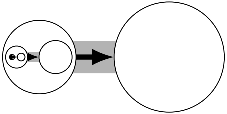

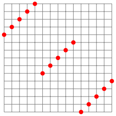

Typical instance $\max_{G \text{ large}}$ #copies of H in G s.t. constraints on G

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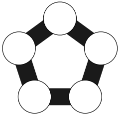

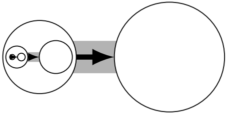

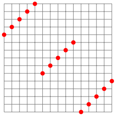

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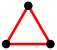
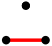
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#$$  $$+ \#$$  $$\approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

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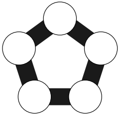

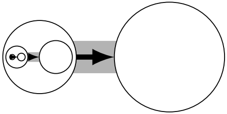

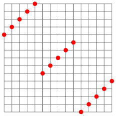

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Duality find specific $\mathbb{E}\left(\cdots\right)^2 \geq 0$ inequalities $\rightarrow$ generates proof

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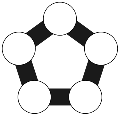
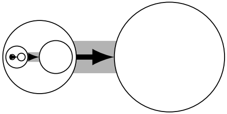
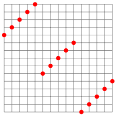




Typical instance $\max_{G \text{ large}}$ #copies of H in G s.t. constraints on G

Idea: use relaxation, optimize over superset $\longrightarrow$ semidefinite prg

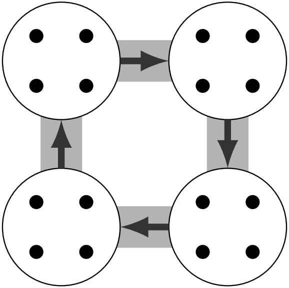
Duality find specific $\mathbb{E}(\cdots)^2 \ge 0$ inequalities $\rightarrow$ generates proof

Such search can automatized and computer assisted (SDP solvers)

The exact value of semi-degree threshold for $\vec{C}_6$

Conjecture (Kelly-Kühn-Osthus, 2010)

Fix $\ell \geq 4, n \geq n_0$. G n-vertex oriented with $\delta^{\pm}(G) > \frac{n}{k_\ell} \Rightarrow \vec{C}_\ell \in G$ where k_ℓ is the smallest integer ≥ 3 that does not divide ℓ



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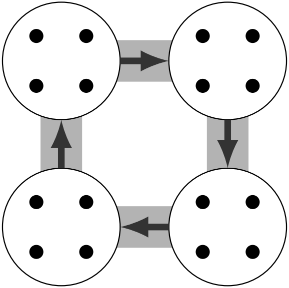

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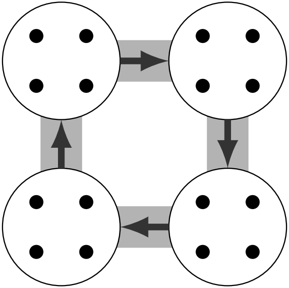

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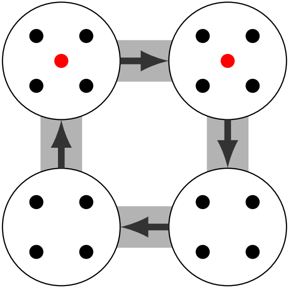

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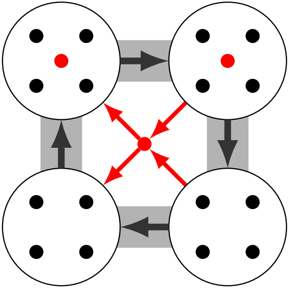

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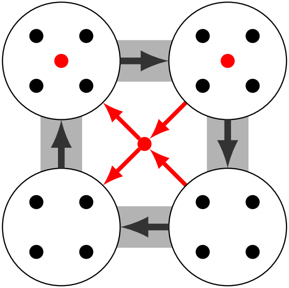

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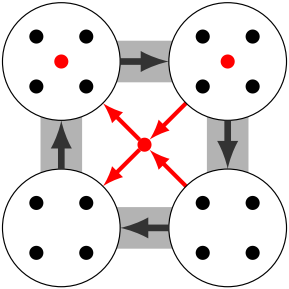
Similarly G_ℓ : $\delta^{\pm}(G_\ell) = \left\lfloor \frac{n}{k_\ell} \right\rfloor + 1$ but no $\vec{C}_\ell$ (except $k_\ell = 4$ & $\ell \equiv 3$)



Semi-degree thresholds for longer oriented cycles C_ℓ

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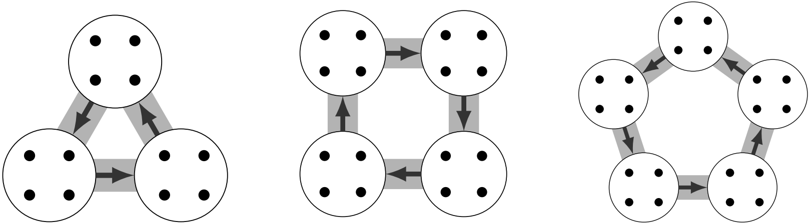

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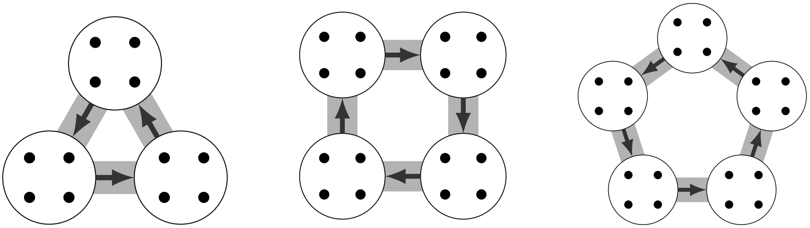
$\delta^\pm(G)$ bound $\Longrightarrow$ arc $u \to w$ & path from u to w of length $\leq k_\ell/2$



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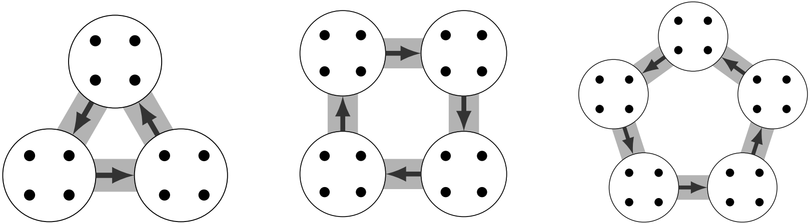

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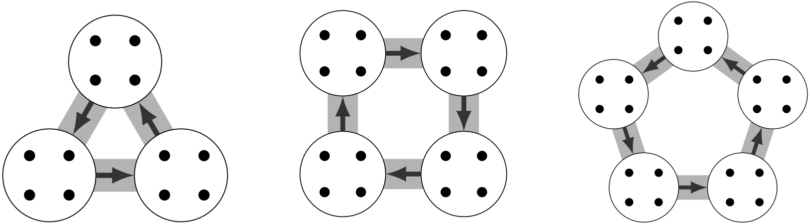

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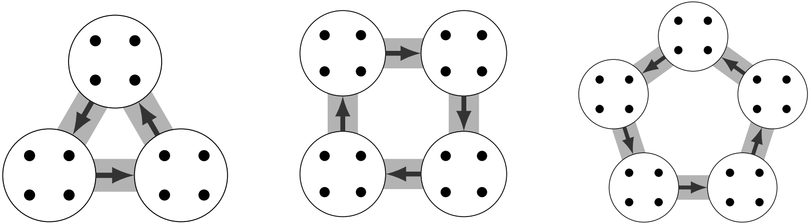

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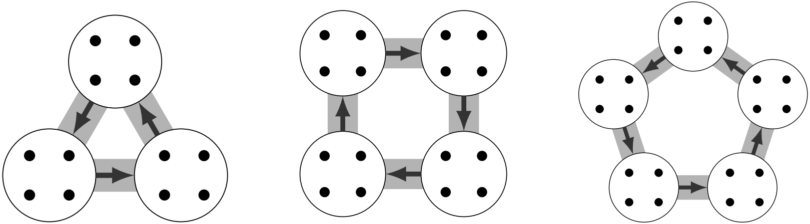
+Flag A for $k_\ell \in \{4, 5\}$ yield $\delta^\pm(G) > \frac{n}{k_\ell} \Rightarrow$ closed walk of length ℓ



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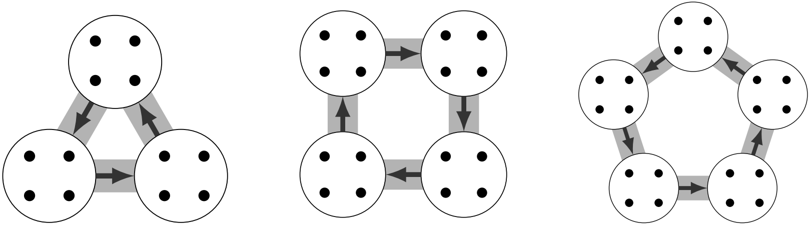

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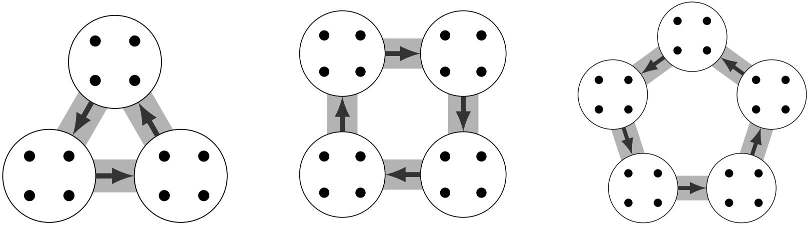

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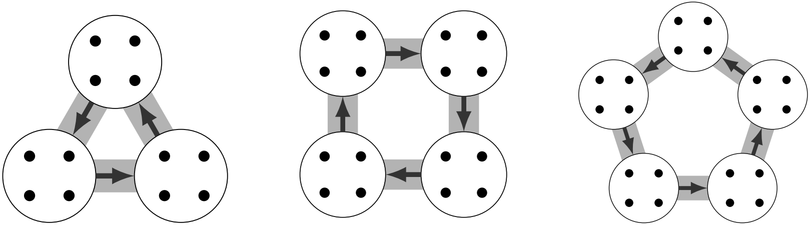

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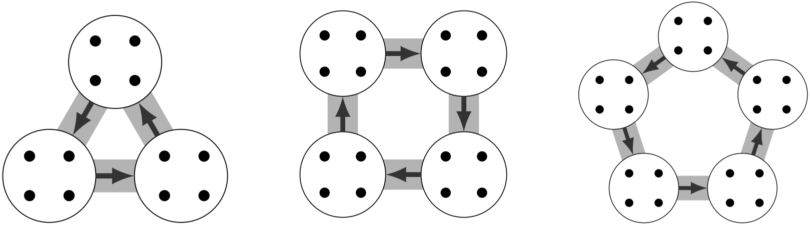

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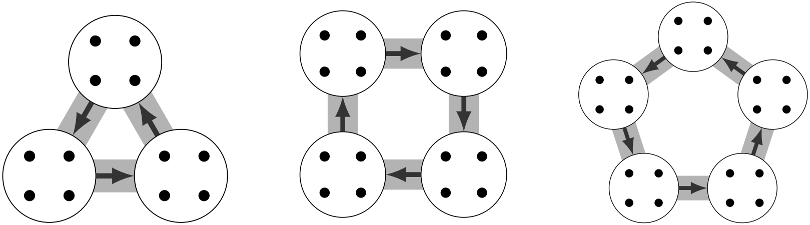

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Thank you for your attention!

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