

On degree thresholds of cycles in oriented graphs

Jan Volec

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joint work with Roman Glebov and Andrzej Grzesik

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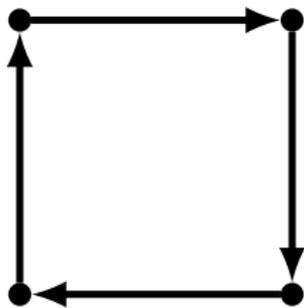
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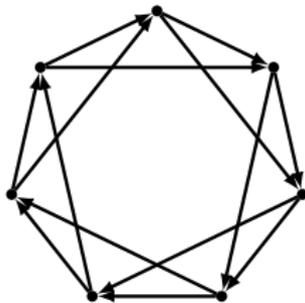
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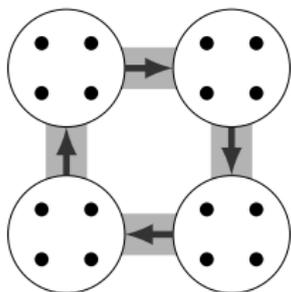
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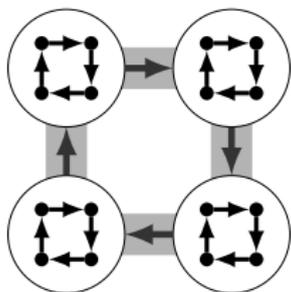
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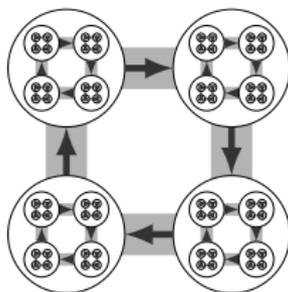
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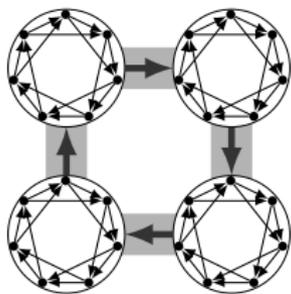
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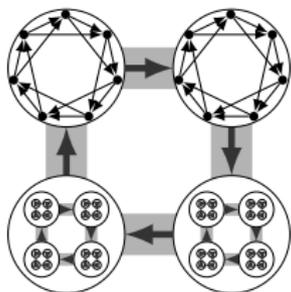
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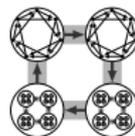
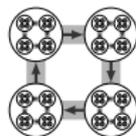
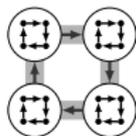
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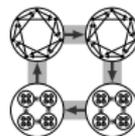
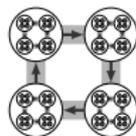
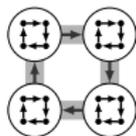
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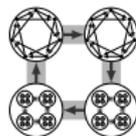
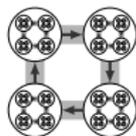
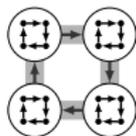
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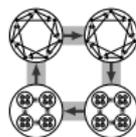
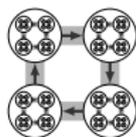
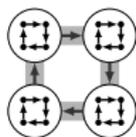
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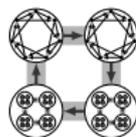
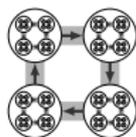
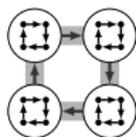
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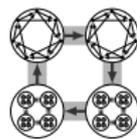
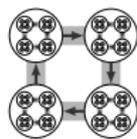
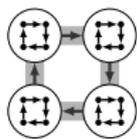
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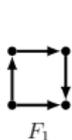
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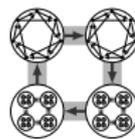
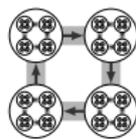
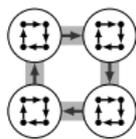


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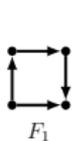


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The C-H conjecture and around: semi-degree thresholds

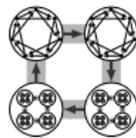
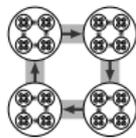
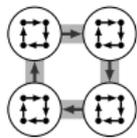
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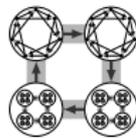
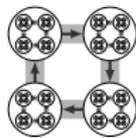
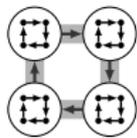


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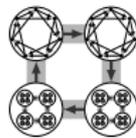
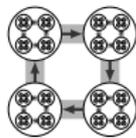
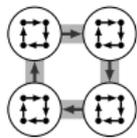
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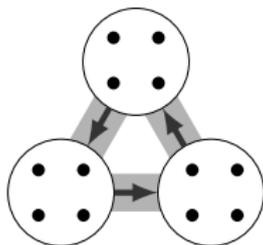
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If ℓ not multiple of 3, then best possible!



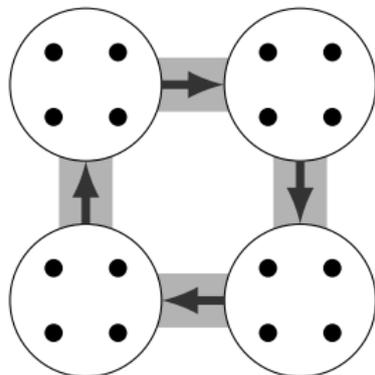
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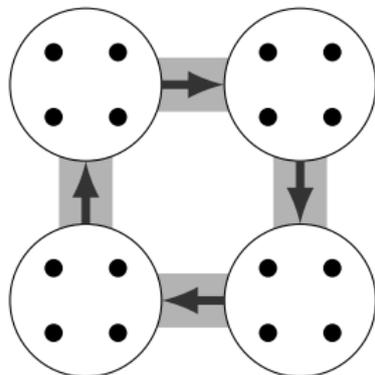


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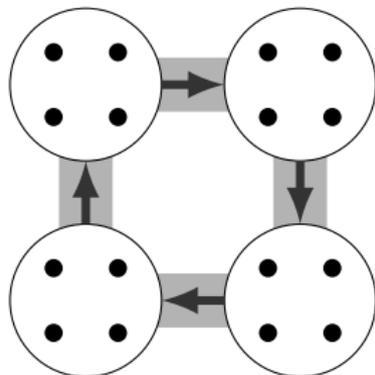
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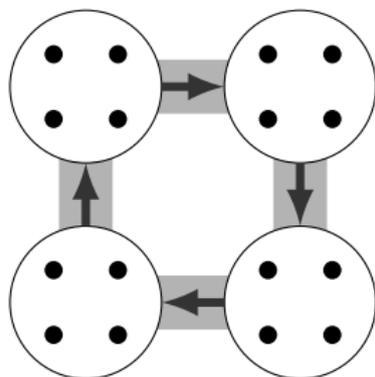
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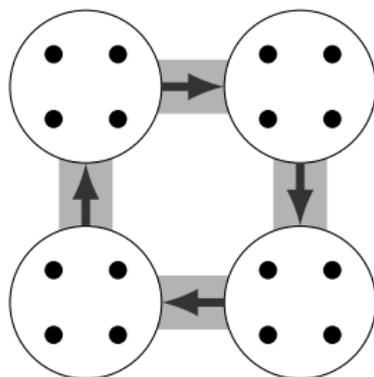
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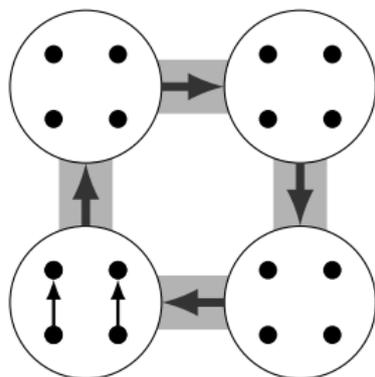
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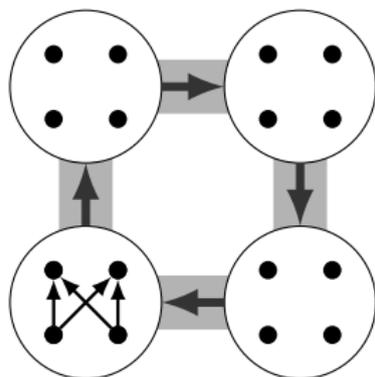
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1st part today: shortest open case $\ell = 6$ i.e δ^\pm threshold $\vec{C}_6 = \frac{1}{4}$



Conjecture of Kelly-Kühn-Osthus on δ^\pm thresholds

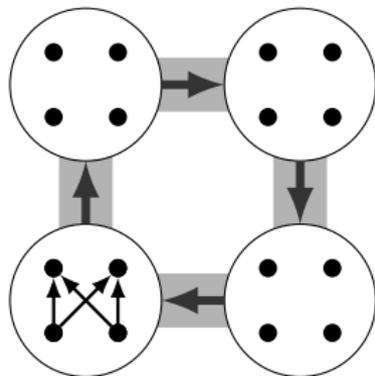
For $\ell \in \mathbb{N} \rightarrow k_\ell :=$ the smallest integer ≥ 3 that does not divide ℓ

Fix $\ell \geq 4, n \geq n_0$. G n -vertex oriented with $\delta^\pm(G) > \frac{n}{k_\ell} \Rightarrow \vec{C}_\ell \in G$

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Stronger conjecture: G n -vertex oriented $\delta^+(G) \gg \frac{n}{k_\ell} \Rightarrow \vec{C}_\ell \in G$?

Sketch of the proof of asymptotic δ^\pm threshold for \vec{C}_6

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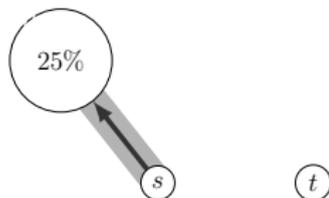
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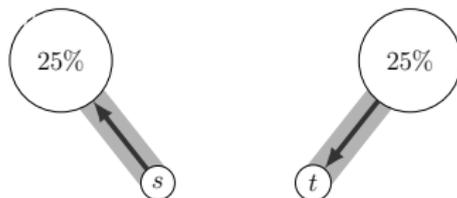
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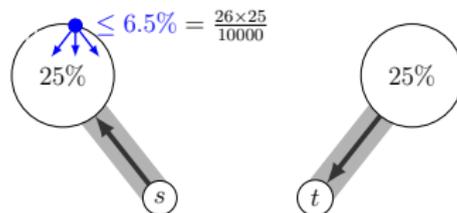
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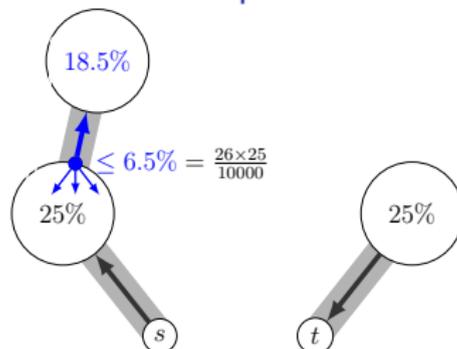
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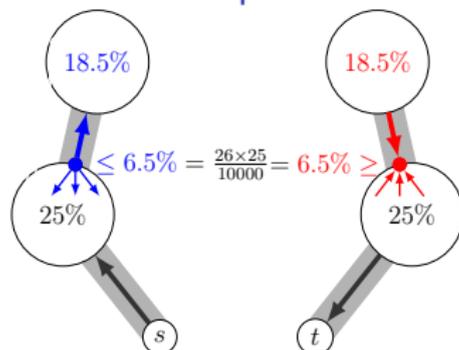
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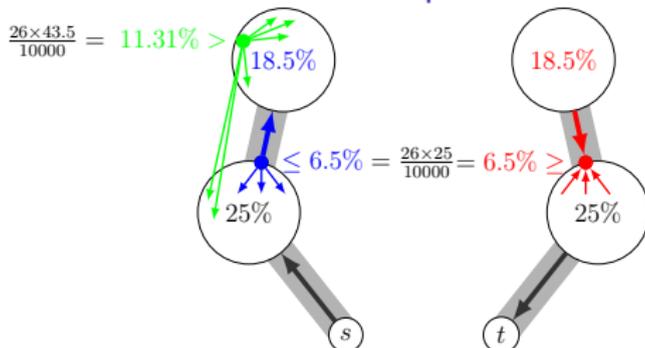
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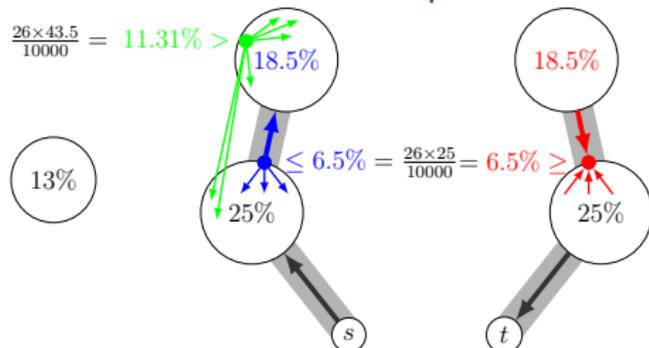
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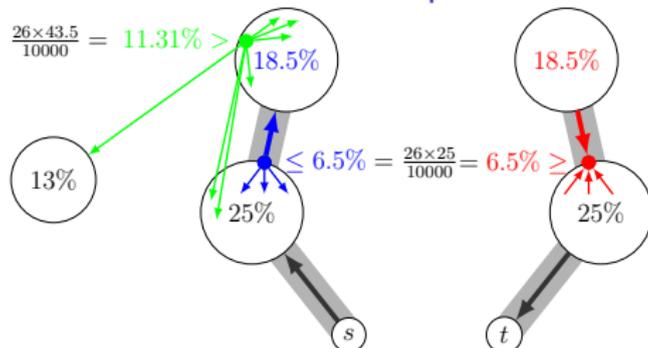
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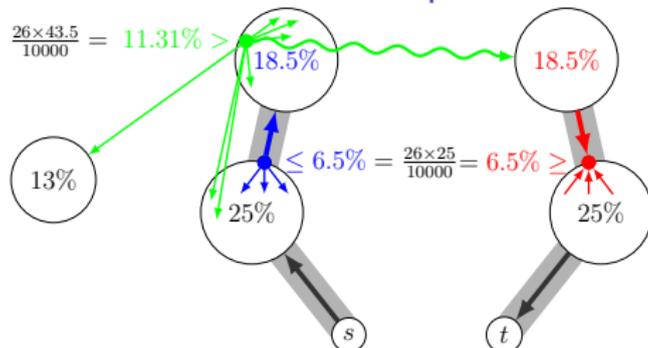
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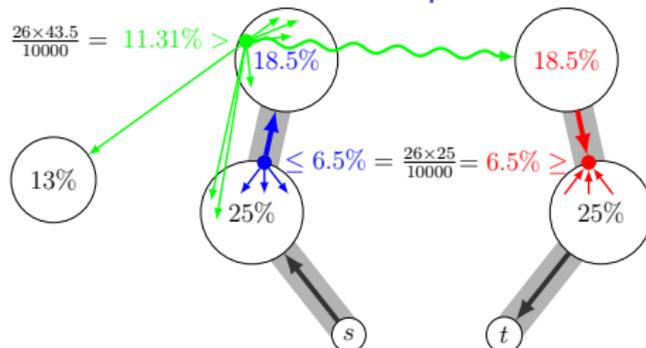
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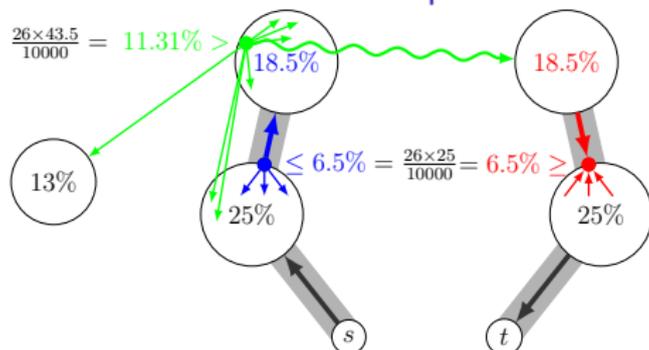
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For oriented path xyz in $N^+(w)$ use $z \xrightarrow{2,3,4,5} w$ path $\implies \vec{C}_3$ or \vec{C}_6

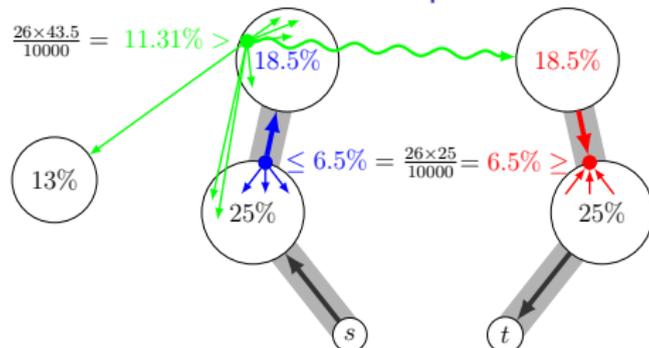
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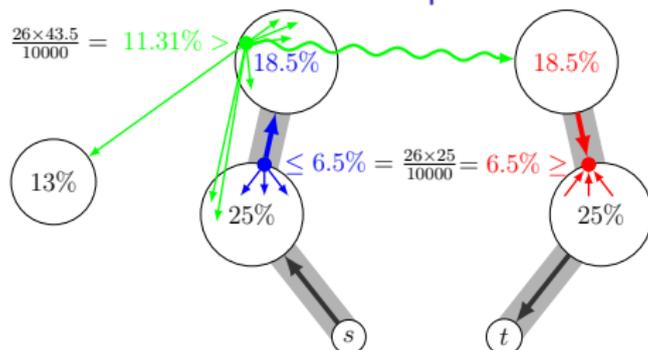
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 Do Corollary again but inside $N^+(s)$ & $N^-(t)$ pick sink (source)

Main tool for $\delta^+ \geq 0.26 \Rightarrow \vec{C}_6$: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

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Analyze (dense) large graphs via relations between small subgraphs

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Analyze (dense) large graphs via relations between small subgraphs

$$\sum_u \left(\frac{n}{2} - \deg(u) \right)^2 - 3 \times \# \triangle + \# \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \approx n \times \left(\frac{n^2}{4} - e(G) \right)$$

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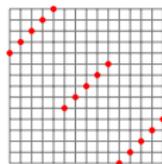
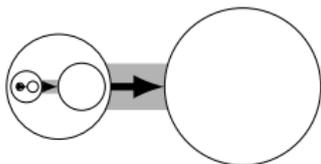
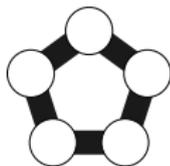
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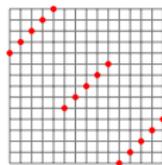
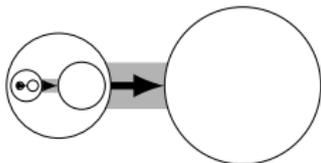
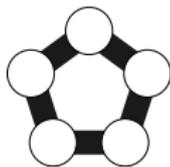
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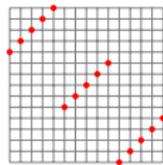
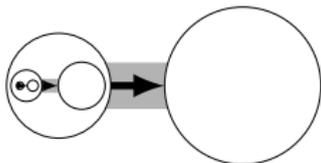
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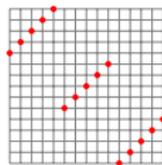
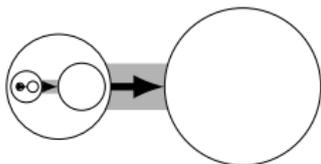
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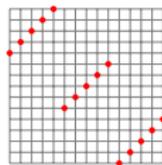
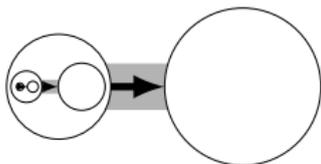
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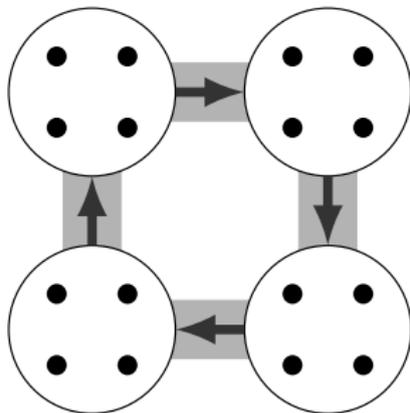
Duality find specific $\mathbb{E}(\dots)^2 \geq 0$ inequalities \rightarrow generates proof

Such search can automatized and computer assisted (SDP solvers)

The exact value of semi-degree threshold for \vec{C}_ℓ

Conjecture (Kelly-Kühn-Osthus, 2010)

Fix $\ell \geq 4, n \geq n_0$. G n -vertex oriented with $\delta^\pm(G) > \frac{n}{k_\ell} \Rightarrow \vec{C}_\ell \in G$
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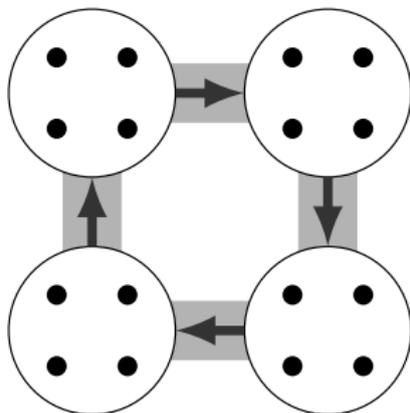


The exact value of semi-degree threshold for \vec{C}_6

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For $\ell = 6$ they conjectured $\delta^\pm(G) \geq \lfloor \frac{n}{4} \rfloor + 1 \Rightarrow \vec{C}_6 \in G$

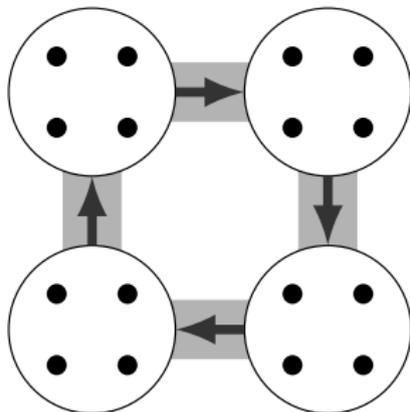


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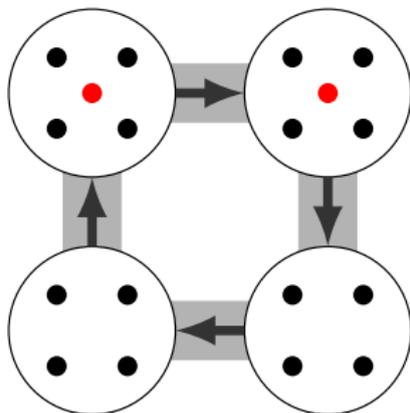


The exact value of semi-degree threshold for \vec{C}_6

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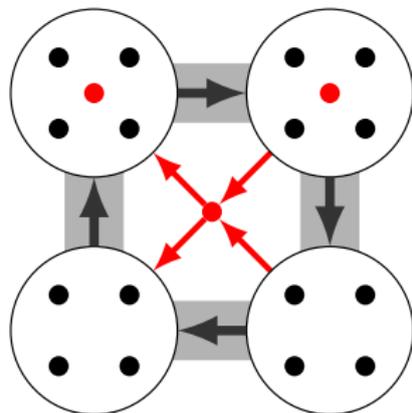


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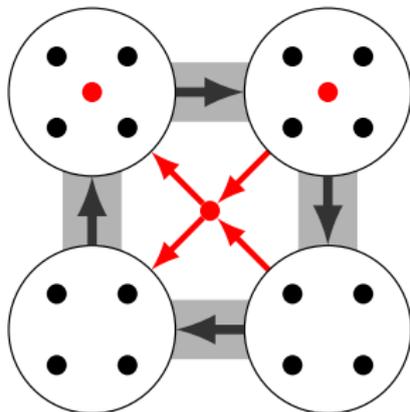
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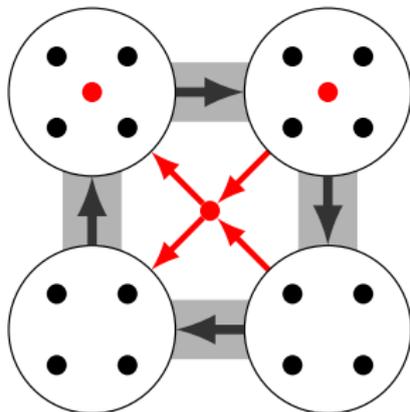
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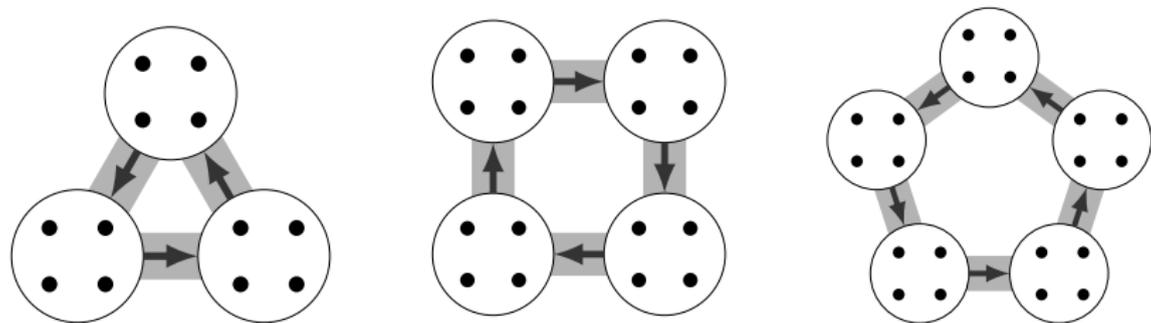
Similarly $\exists G_\ell : \delta^\pm(G_\ell) > \lfloor \frac{n}{k_\ell} \rfloor$ but no \vec{C}_ℓ (except $\ell = 12a + 3$)



Semi-degree thresholds for longer oriented cycles

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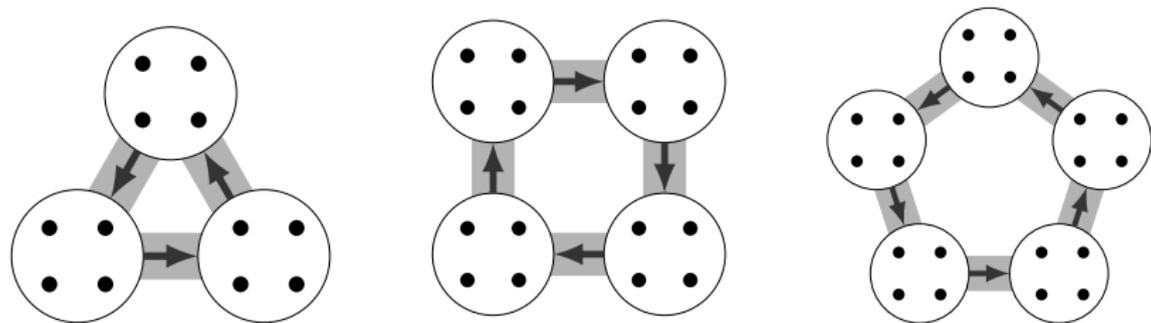


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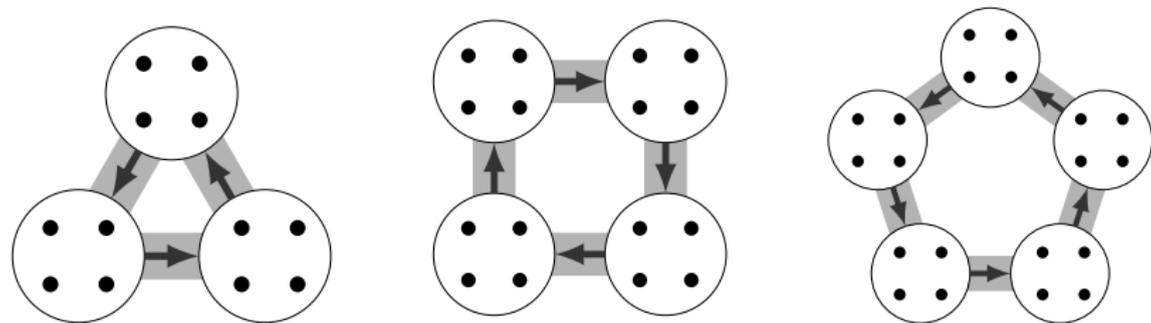
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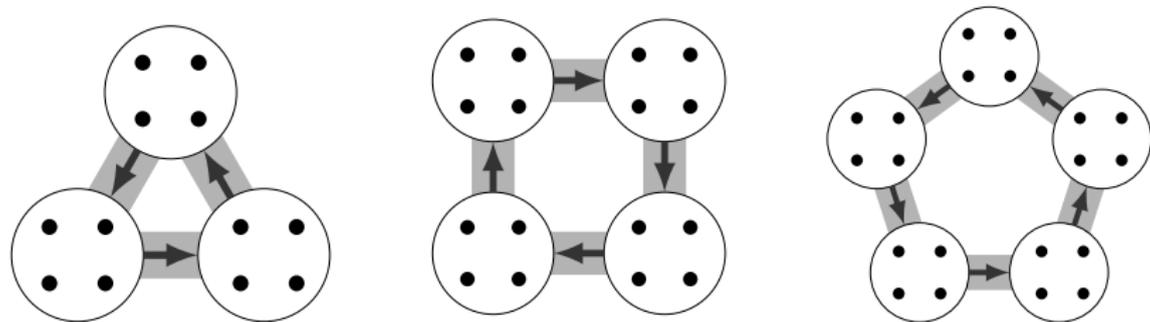
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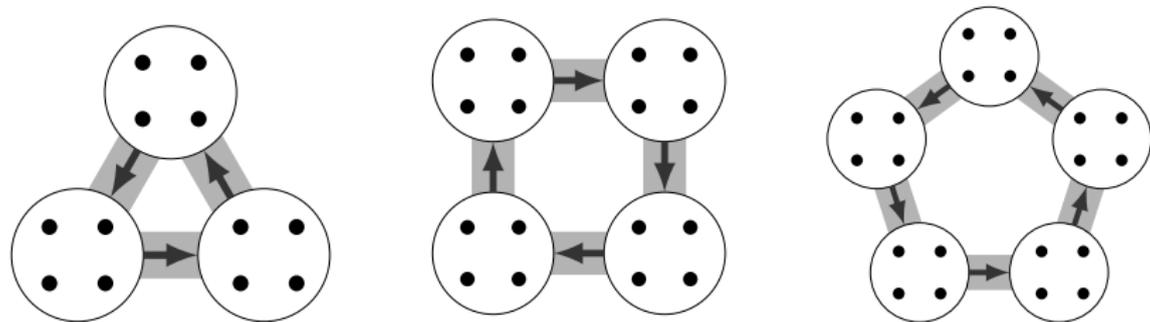
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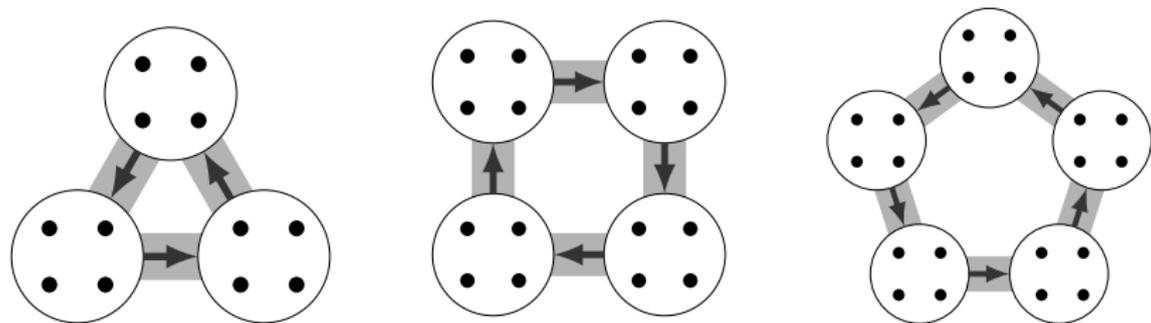
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Exact semi-degree threshold for \vec{C}_ℓ

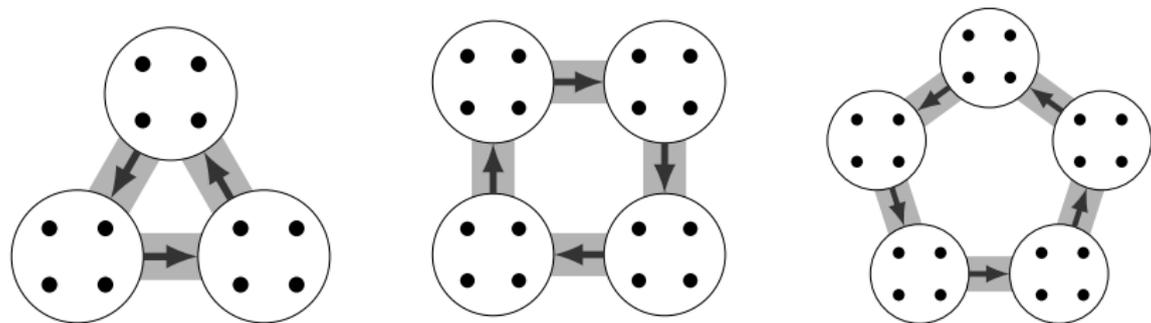
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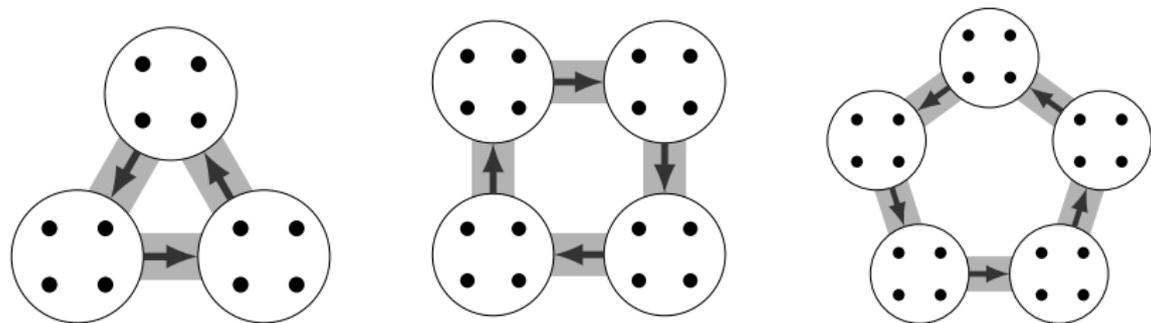
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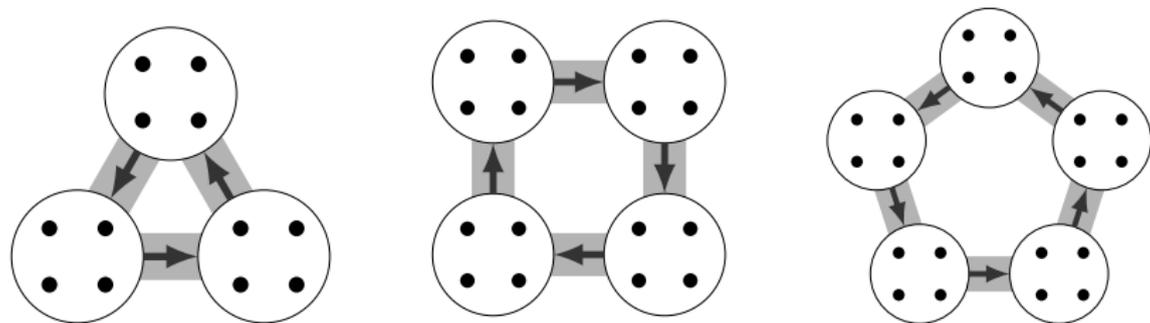


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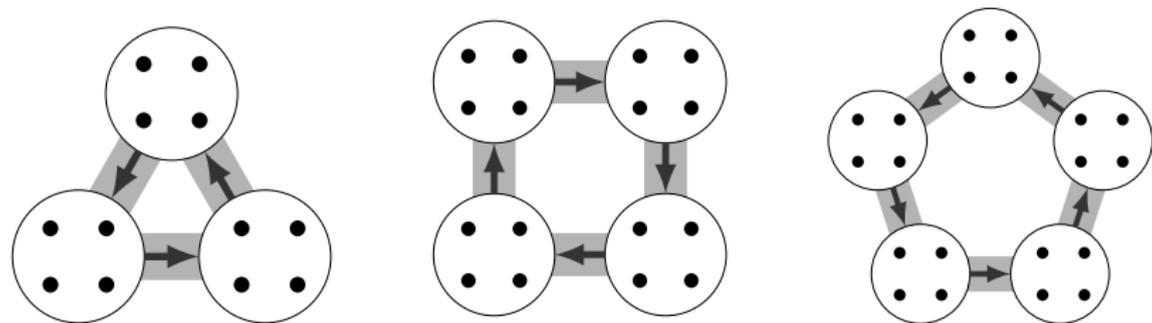
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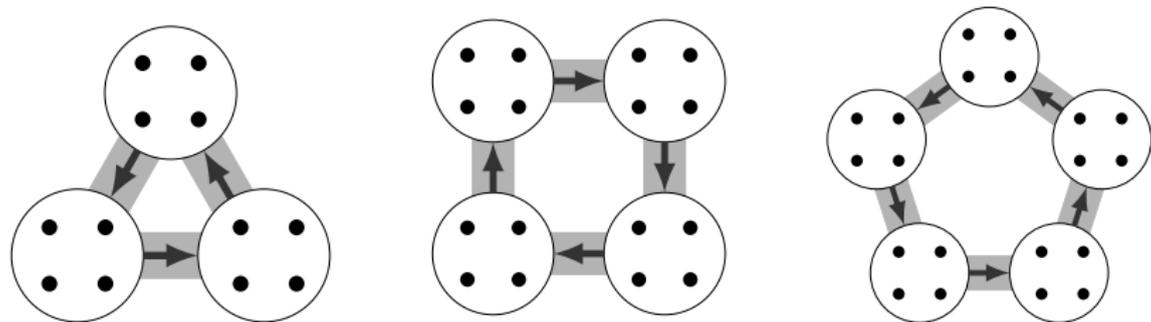
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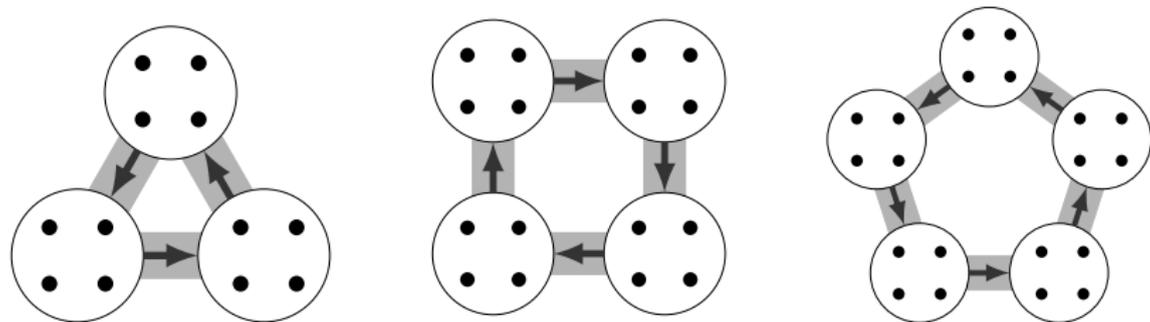
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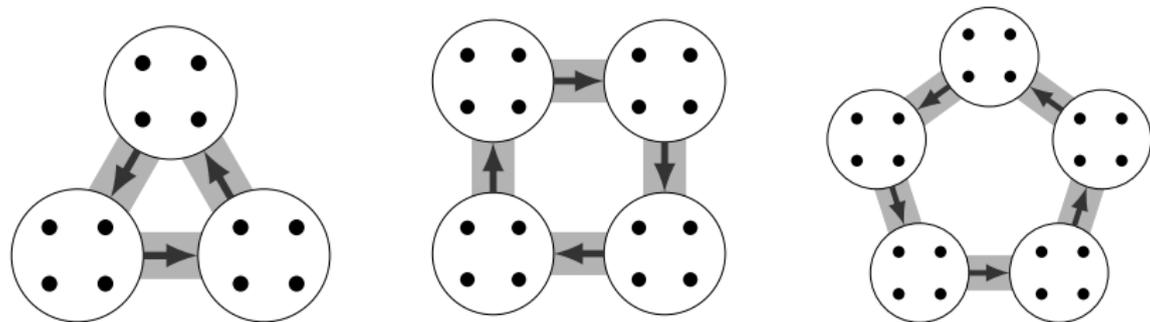
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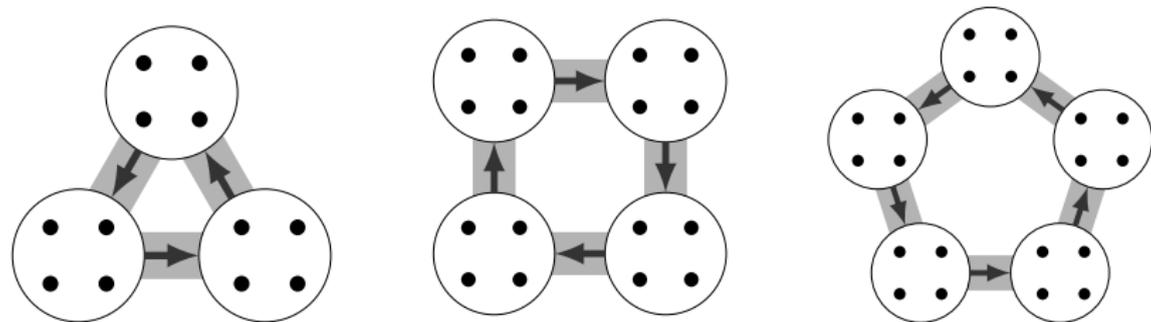
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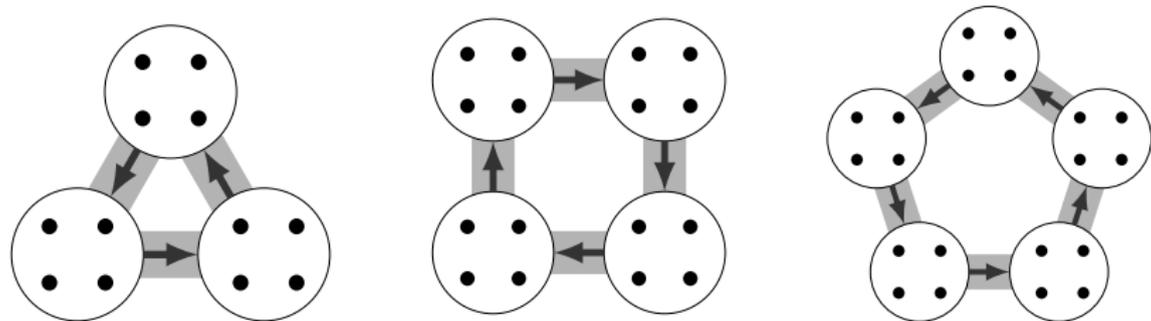
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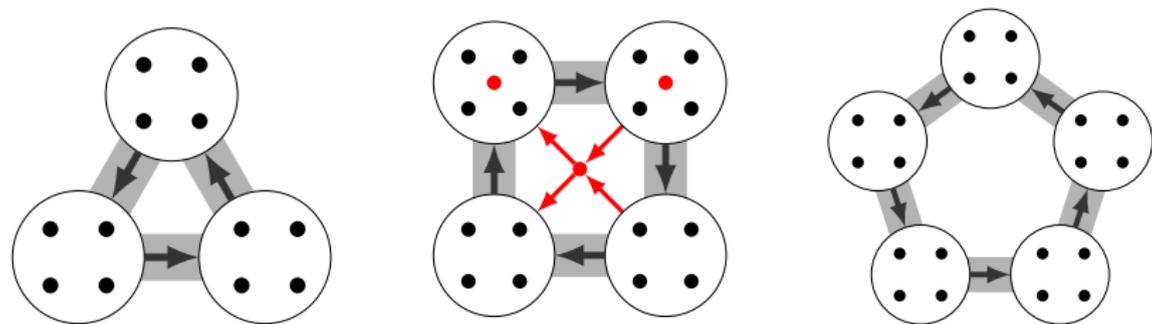
Best $\ell \neq 12a + 3$. For $\ell = 12a + 3$, then $\delta^\pm(G) \geq \frac{n}{k_\ell} + \frac{1}{4} \Rightarrow \vec{C}_\ell \in G$



Conclusion

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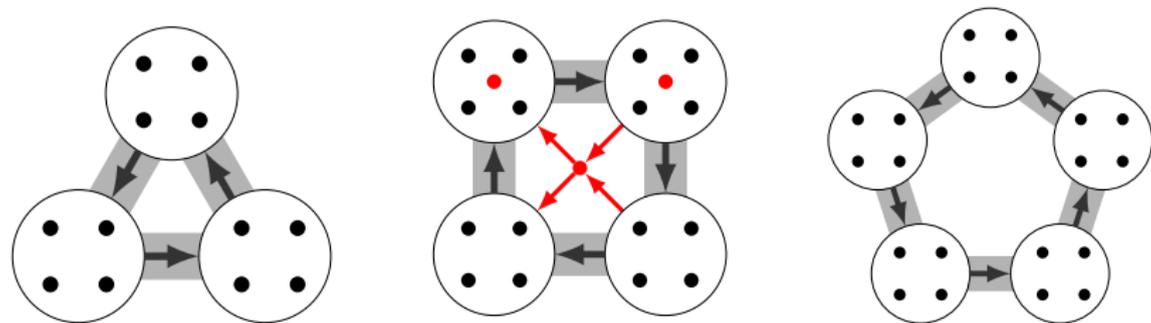
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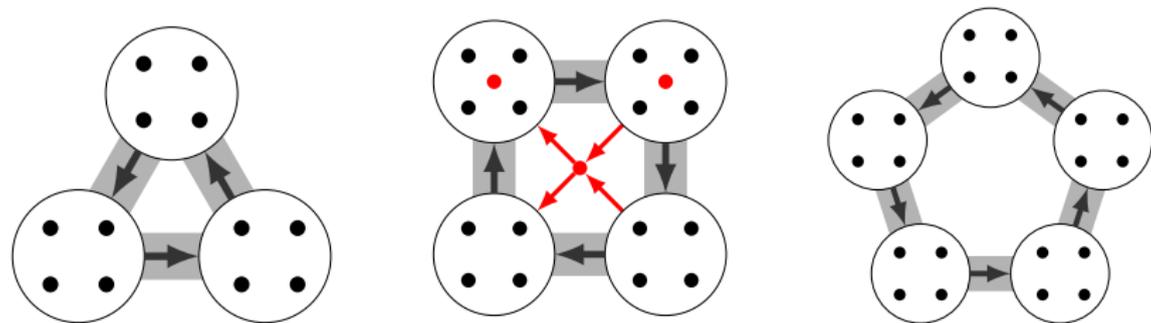
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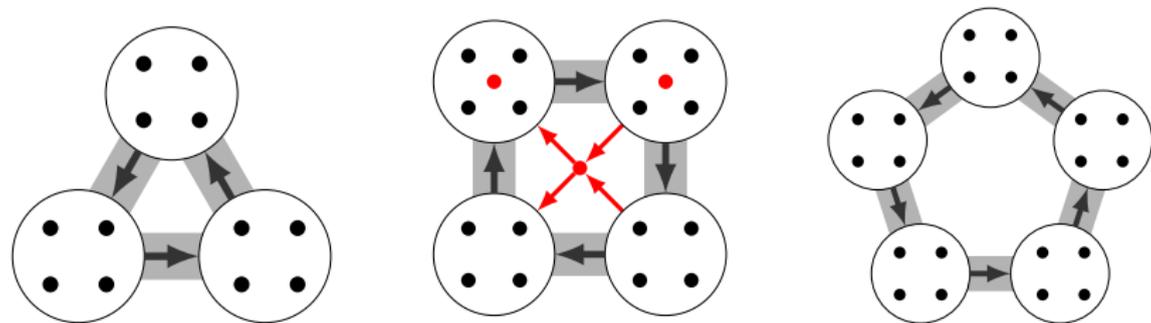
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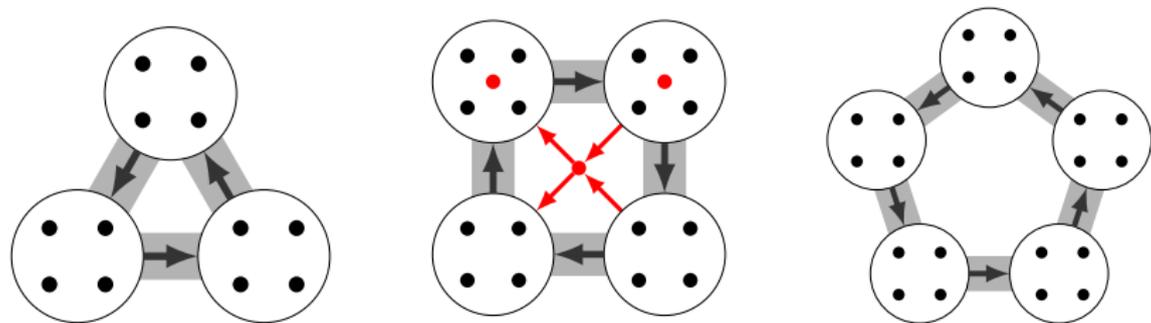
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