Forcing quasirandomness in permutations

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Based on a joint work with T. Chan, D. Kráľ, J. Noel, Y. Pehova, and M.Sharifzadeh.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 800607.

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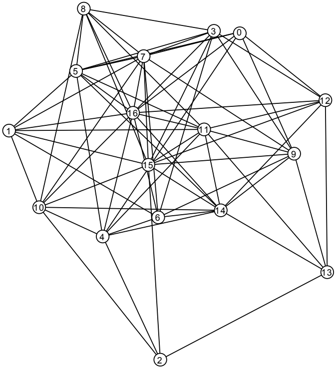

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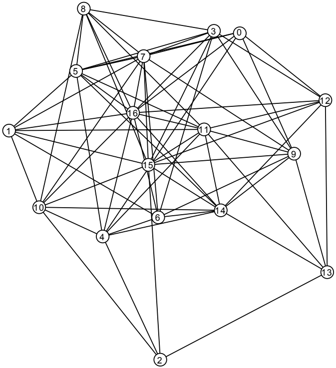
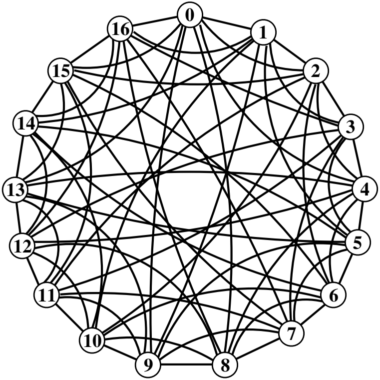


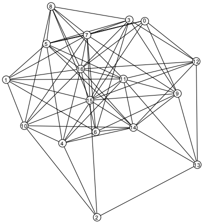


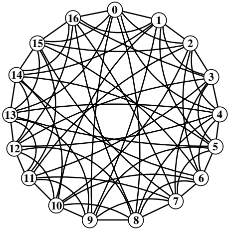
Quasi-random / pseudorandom graphs

Late 80's: Chung-Graham-Wilson extending Rödl and Thomasson:

Large n-vertex graph $G = (V, E)$ of density p is quasi-random if:

- $\forall X \subseteq V$, $G[X]$ has $\approx p \cdot \binom{|X|}{2}$ edges
- $\forall A, B \subseteq V$, # of edges between A and B is $\approx p \cdot |A||B|$
- $\forall F$, # of *labelled non-induced* F in G is $\approx p^{e(F)} \cdot n^{v(F)}$
- For $F = C_4$, # of labelled non-induced $F \subset G$ is $\approx p^4 \cdot n^4$
- $\sum_{u,w \in V} \left| \operatorname{codeg}(u, w) - p^2 \cdot n \right| = o\left(n^3\right)$
- $\lambda_2(G) = o(n)$





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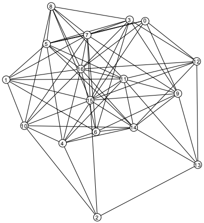

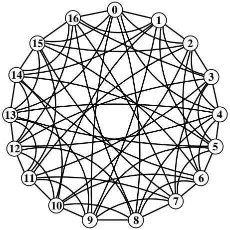

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Smallest open case for both conjectures: $B = K_{5,5} - C_{10}$

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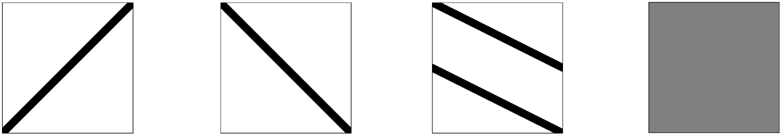

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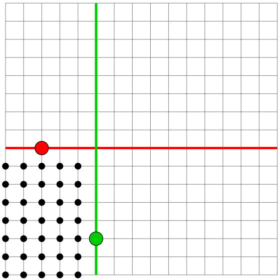

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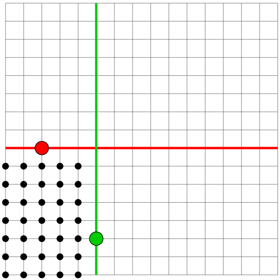

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Theorem (Král'-Pikhurko '13) $\forall \tau \in S_5 : d(\tau, \mu) = \frac{1}{120} \Rightarrow \mu$ uniform

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Original proof uses density $d(\tau, \mu)$ of all 24 permutations $\tau \in S_4$

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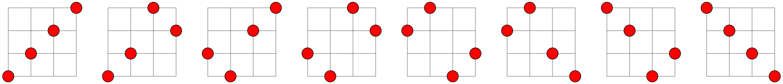

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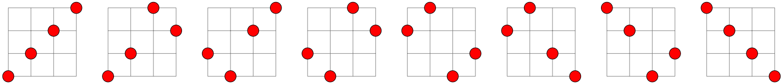

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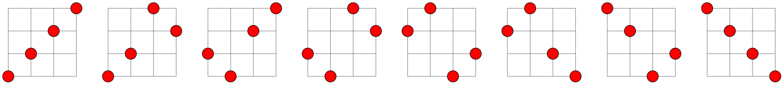

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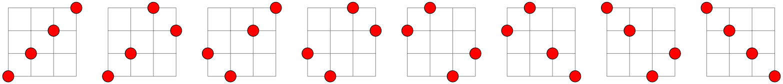

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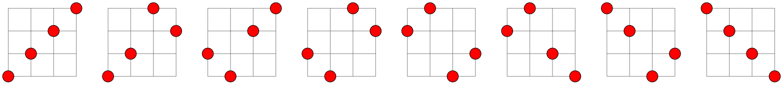

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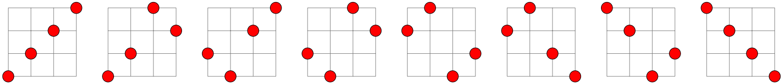

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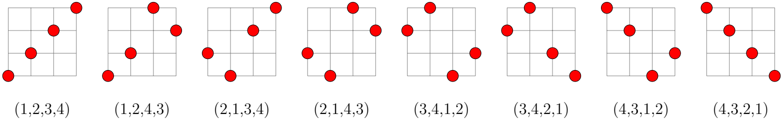

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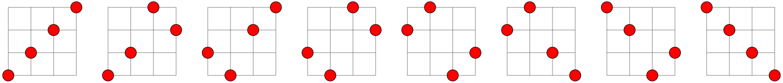

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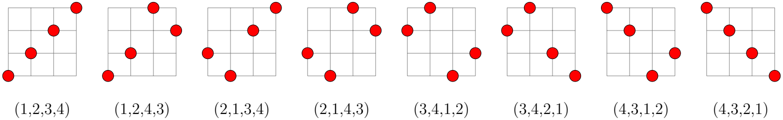

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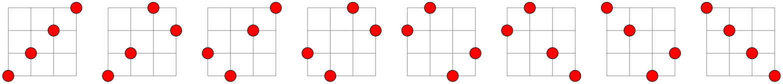

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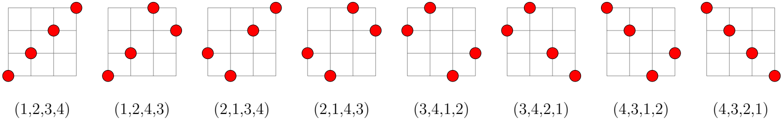

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