Transversal and colorful versions of Mantel's theorem

Jan Volec

MSCA global fellow at Emory University & Universität Hamburg

Based on joint works with E. Culver, B. Lidický, F. Pfender and S. Norin

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 800607.

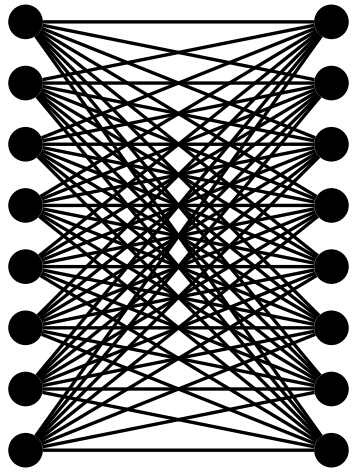
Intro to extremal graph theory – Mantel's theorem

What 16-vertex G has maximum number of  but yet no  ?

Intro to extremal graph theory – Mantel's theorem

What 16-vertex G has maximum number of  but yet no  ?

A: complete bipartite graph with the parts of size 8 (#edges $= 64$)

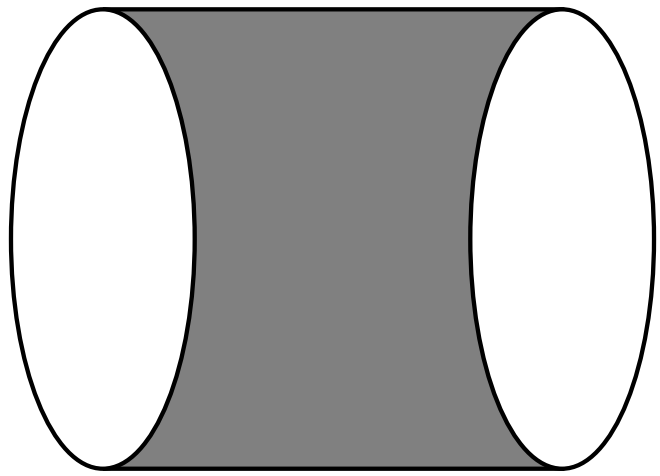


Intro to extremal graph theory – Mantel's theorem

What 16-vertex G has maximum number of  but yet no  ?

A: complete bipartite graph with the parts of size 8 (#edges $= 64$)

Mantel (1907): Every n-vertex graph with no  has $\leq \frac{n^2}{4}$ edges

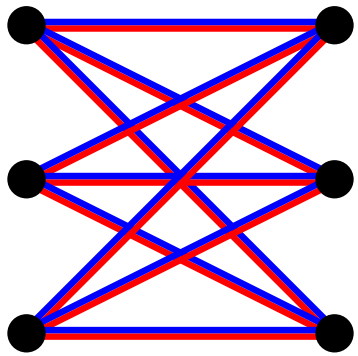


Mantel's theorem[2]

Given graphs G_1 and G_2 on $V = [n]$, what edge-bounds guarantee non-monochromatic triangle?

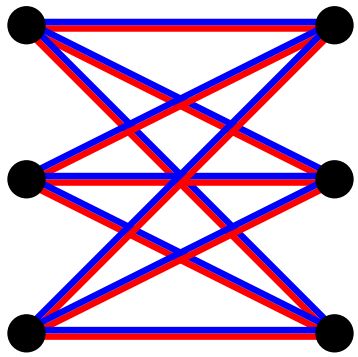
Mantel's theorem^2

Given graphs G_1 and G_2 on $V = [n]$, what edge-bounds guarantee non-monochromatic triangle? Must be $> n^2/4$



Mantel's theorem[2]

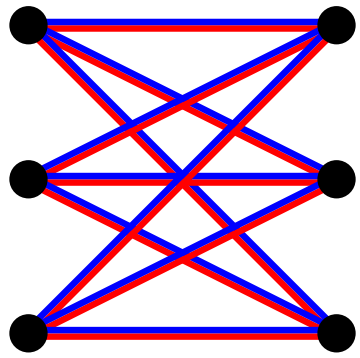
Given graphs G_1 and G_2 on $V = [n]$, what edge-bounds guarantee non-monochromatic triangle? Must be $> n^2/4$



Proposition: No non-mono triangle $\Longrightarrow e(G_1) + e(G_2) \leq n^2/2$

Mantel's theorem^2

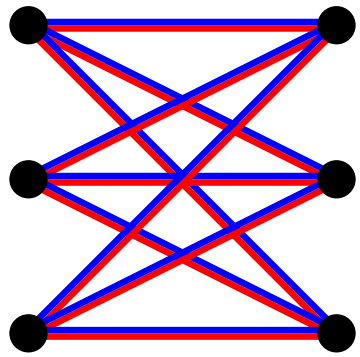
Given graphs G_1 and G_2 on $V = [n]$, what edge-bounds guarantee non-monochromatic triangle? Must be $> n^2/4$



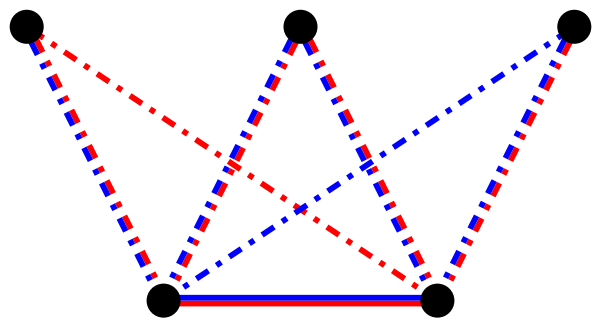
Proposition: No non-mono triangle $\Longrightarrow e(G_1) + e(G_2) \leq n^2/2$

Mantel's theorem[2]

Given graphs G_1 and G_2 on $V = [n]$, what edge-bounds guarantee non-monochromatic triangle? Must be $> n^2/4$

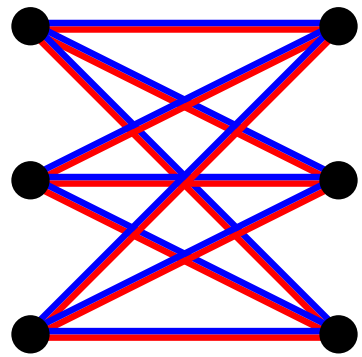


Proposition: No non-mono triangle $\Longrightarrow e(G_1) + e(G_2) \leq n^2/2$

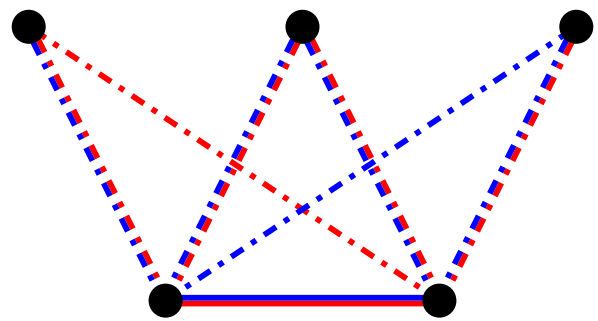


Mantel's theorem[2]

Given graphs G_1 and G_2 on $V = [n]$, what edge-bounds guarantee non-monochromatic triangle? Must be $> n^2/4$



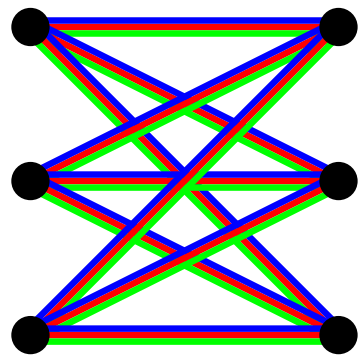
Proposition: No non-mono triangle $\Longrightarrow e(G_1) + e(G_2) \leq n^2/2$



$\Longrightarrow e(G_1) + e(G_2) \leq (n-2)^2/2 + 2(n-2) + 2 = n^2/2$ $\square$

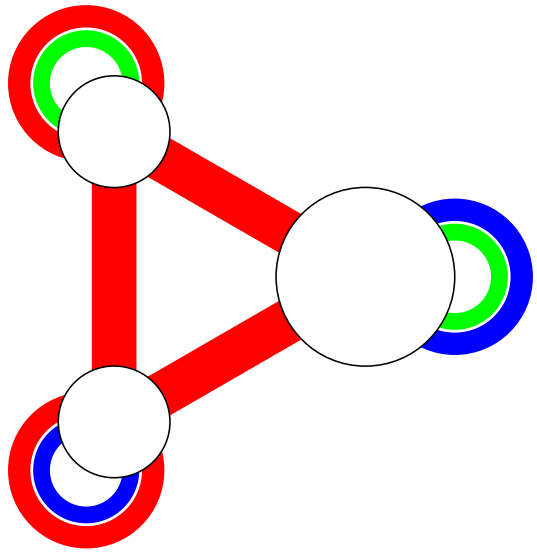
Mantel's theorem[3]

Given graphs G_1, G_2, G_3 on $V = [n]$, what edge-bounds guarantee transversal triangle? Must be $> n^2/4$



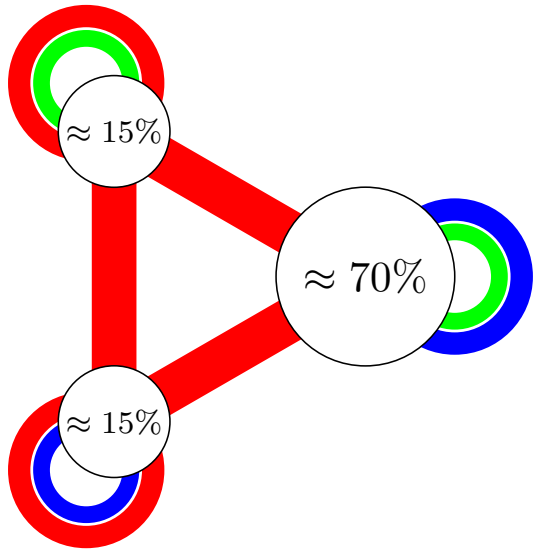
Mantel's theorem[3]

Given graphs G_1, G_2, G_3 on $V = [n]$, what edge-bounds guarantee transversal triangle? Must be $\gg n^2/4$



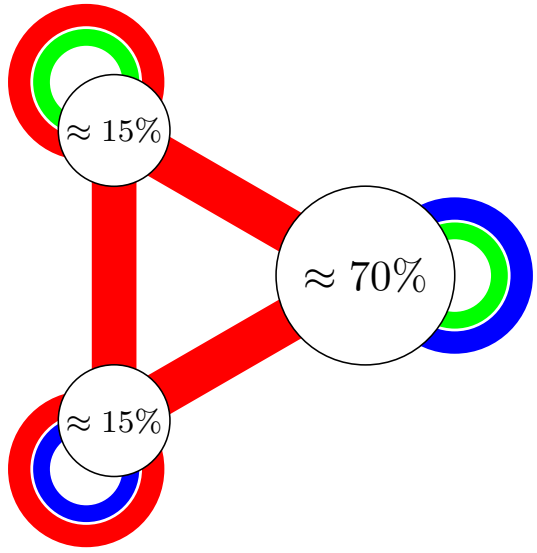
Mantel's theorem[3]

Given graphs G_1, G_2, G_3 on $V = [n]$, what edge-bounds guarantee transversal triangle? Must be $> \left(52 - 4 \cdot \sqrt{7}\right) \cdot n^2/162 \approx .256n^2$



Mantel's theorem[3]

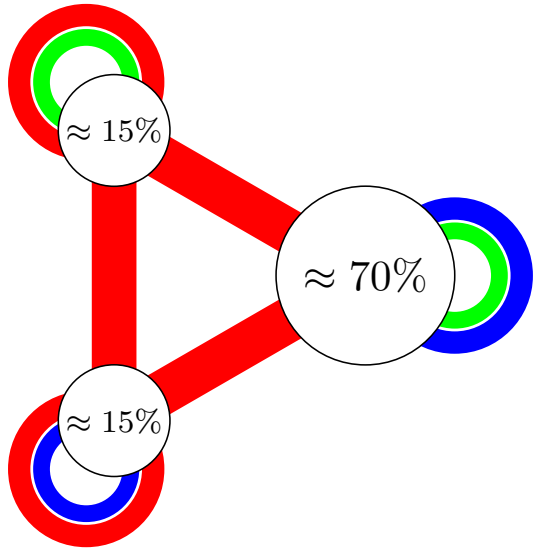
Given graphs G_1, G_2, G_3 on $V = [n]$, what edge-bounds guarantee transversal triangle? Must be $> \left(52 - 4 \cdot \sqrt{7}\right) \cdot n^2/162 \approx .256n^2$



Conjecture (Aharoni-DeVos): this construction is best possible.

Mantel's theorem[3]

Given graphs G_1, G_2, G_3 on $V = [n]$, what edge-bounds guarantee transversal triangle? Must be $> \left(52 - 4 \cdot \sqrt{7}\right) \cdot n^2/162 \approx .256n^2$



Conjecture (Aharoni-DeVos): this construction is best possible.

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
Yes, if all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$,
$\Longrightarrow$ transversal triangle.

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

Analyze (dense) large graphs via relations between small subgraphs

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

Analyze (dense) large graphs via relations between small subgraphs

$$\sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\overset{\bullet}{\text{—}} \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

Analyze (dense) large graphs via relations between small subgraphs

$$0 \leq \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\overset{\bullet}{\text{—}} \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

for every n-vertex G with $\#\,\triangle = 0$

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

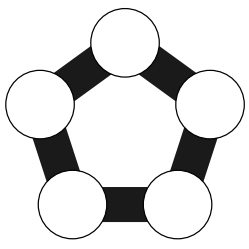
Analyze (dense) large graphs via relations between small subgraphs

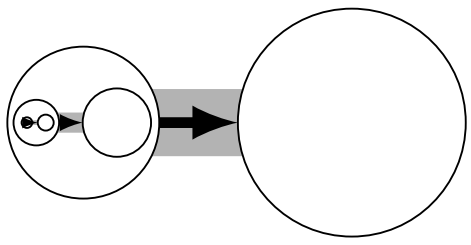
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\text{P}_3 \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

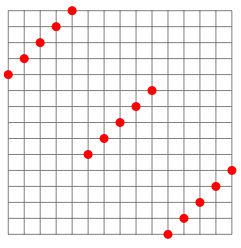
for every n-vertex G with $\#\,\triangle = 0$

Powerful tool, shed light to many open problems

max C_5 subj. to $\triangle = 0$ max Λ in orient graph min $\tau(1234) + \tau(4321)$ in perm







Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

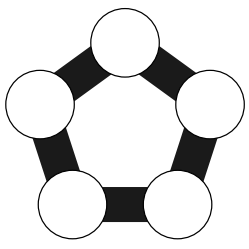
Analyze (dense) large graphs via relations between small subgraphs

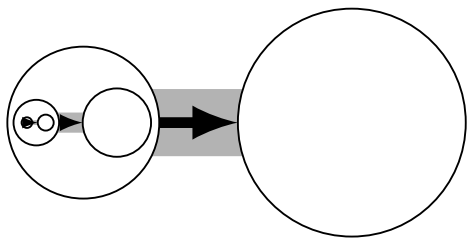
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\text{P}_3 \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

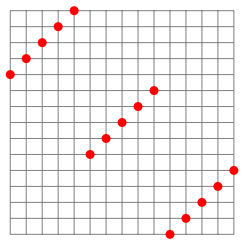
for every n-vertex G with $\#\,\triangle = 0$

Powerful tool, shed light to many open problems

max C_5 subj. to $\triangle = 0$ max P_3 in orient graph min $1234 + 4321$ in perm







Typical instance $\max_{G \text{ large}}$ #copies of H in G s.t. constraints on G

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

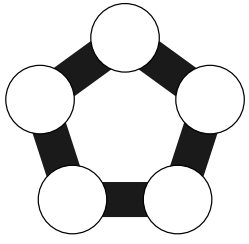
Analyze (dense) large graphs via relations between small subgraphs

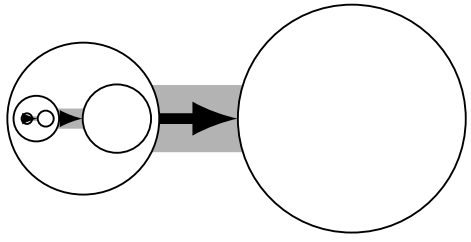
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\text{P}_3 \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

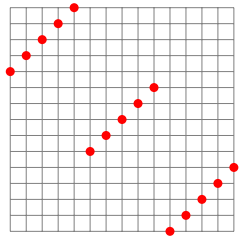
for every n-vertex G with $\#\,\triangle = 0$

Powerful tool, shed light to many open problems

max C_5 subj. to $\triangle = 0$ max $\overrightarrow{\text{P}_3}$ in orient graph min $\overset{1234}{\underset{4321}{+}}$ in perm







Typical instance $\max_{G \text{ large}}$ #copies of H in G s.t. constraints on G

Idea: use relaxation, optimize over superset $\longrightarrow$ semidefinite prg

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

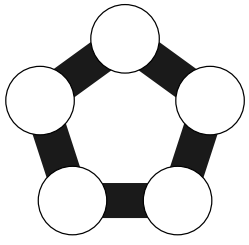
Analyze (dense) large graphs via relations between small subgraphs

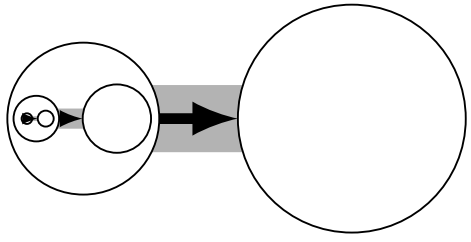
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\text{P}_3 \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

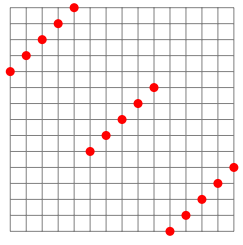
for every n-vertex G with $\#\,\triangle = 0$

Powerful tool, shed light to many open problems

max C_5 subj. to $\triangle = 0$ max P_3 in orient graph min $\tau(1234) + \tau(4321)$ in perm







Typical instance $\max_{G \text{ large}}$ #copies of H in G s.t. constraints on G

Idea: use relaxation, optimize over superset $\longrightarrow$ semidefinite prg

Duality find specific $\mathbb{E}(\cdots)^2 \ge 0$ inequalities $\rightarrow$ generates proof

Main tool: Flag Algebra framework

Razborov (2010): systematic approach to extremal combinatorics

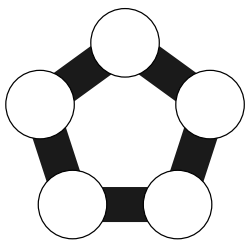
Analyze (dense) large graphs via relations between small subgraphs

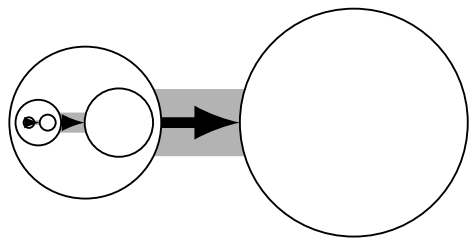
$$0 \le \sum_u \left(\frac{n}{2} - \deg(u)\right)^2 - 3 \times \#\,\triangle + \#\,\text{edge} \approx n \times \left(\frac{n^2}{4} - e(G)\right)$$

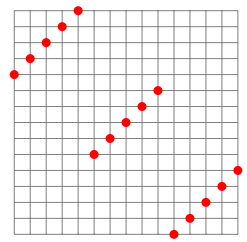
for every n-vertex G with $\#\,\triangle = 0$

Powerful tool, shed light to many open problems

max C_5 subj. to $\triangle = 0$ max $\overrightarrow{P_3}$ in orient graph min $\tau(1234) + \tau(4321)$ in perm







Typical instance $\max_{G \text{ large}}$ #copies of H in G s.t. constraints on G

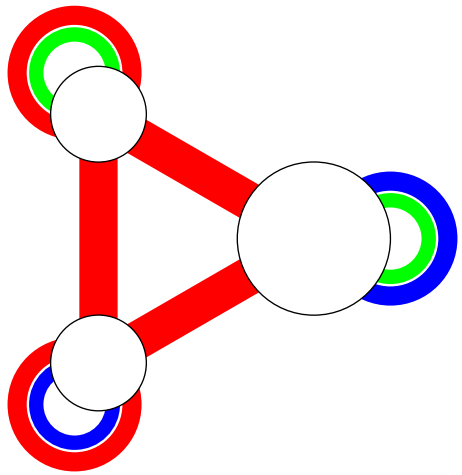
Idea: use relaxation, optimize over superset $\longrightarrow$ semidefinite prg

Duality find specific $\mathbb{E}(\cdots)^2 \ge 0$ inequalities $\rightarrow$ generates proof

Such search can automatized and computer assisted (SDP solvers)

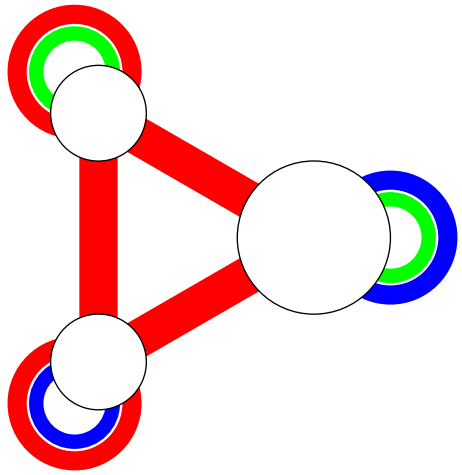
Flag algebras for Aharoni-DeVos conjecture

If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.



Flag algebras for Aharoni-DeVos conjecture

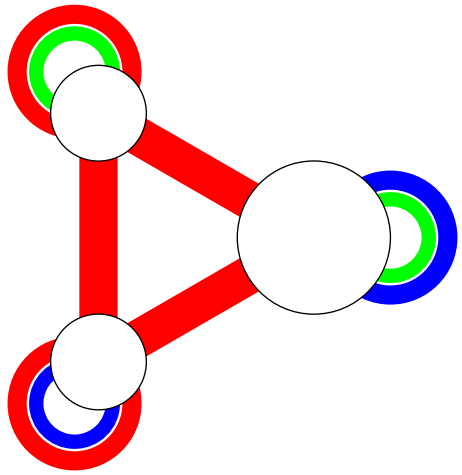
If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.



Plane semidefinite method $\longrightarrow$ seems to work . . . (but huge SDP)

Flag algebras for Aharoni-DeVos conjecture

If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.

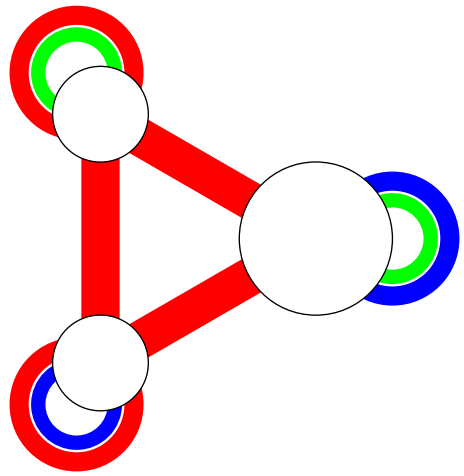


Plane semidefinite method $\longrightarrow$ seems to work . . . (but huge SDP)

"Compression": factorize $(G_1, G_2, G_3) \to \{G_1, G_2, G_3\}$ (color-blind)

Flag algebras for Aharoni-DeVos conjecture

If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.



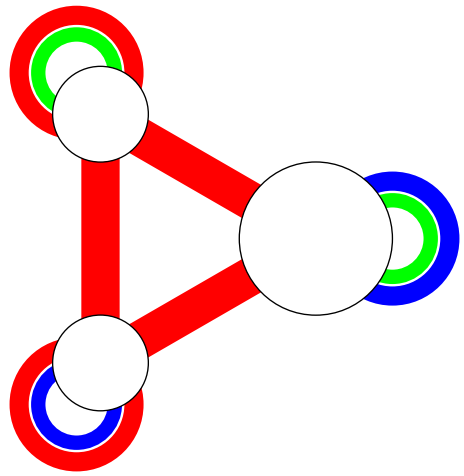
Plane semidefinite method $\longrightarrow$ seems to work . . . (but huge SDP)

"Compression": factorize $(G_1, G_2, G_3) \to \{G_1, G_2, G_3\}$ (color-blind)

How to blindly express RED = BLUE = GREEN assumption?

Flag algebras for Aharoni-DeVos conjecture

If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.



Plane semidefinite method $\longrightarrow$ seems to work . . . (but huge SDP)

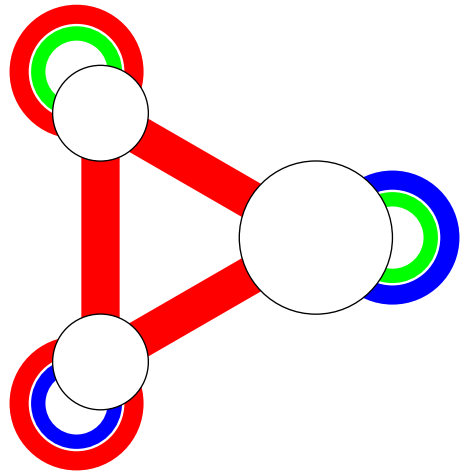
"Compression": factorize $(G_1, G_2, G_3) \to \{G_1, G_2, G_3\}$ (color-blind)

How to blindly express RED = BLUE = GREEN assumption?

Write second moments: RED $\times$ NON-RED = 2·RED $\times$ RED

Flag algebras for Aharoni-DeVos conjecture

If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.



Plane semidefinite method $\longrightarrow$ seems to work . . . (but huge SDP)

"Compression": factorize $(G_1, G_2, G_3) \to \{G_1, G_2, G_3\}$ (color-blind)

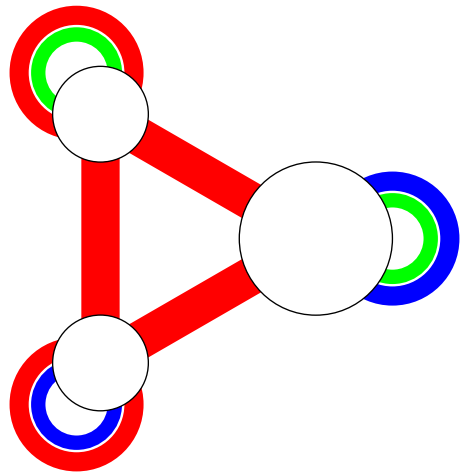
How to blindly express RED = BLUE = GREEN assumption?

Write second moments: RED $\times$ NON-RED = 2·RED $\times$ RED

$\Longrightarrow$ . . . 1.5M of semidefinite inequalities certifying the bound

Flag algebras for Aharoni-DeVos conjecture

If all three graphs G_1, G_2, G_3 on $[n]$ has $\gg \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} \approx .256n^2$, then there is a transversal triangle.



Plane semidefinite method $\longrightarrow$ seems to work . . . (but huge SDP)

"Compression": factorize $(G_1, G_2, G_3) \to \{G_1, G_2, G_3\}$ (color-blind)

How to blindly express RED = BLUE = GREEN assumption?

Write second moments: RED $\times$ NON-RED = 2·RED $\times$ RED

$\Longrightarrow$ . . . 1.5M of semidefinite inequalities certifying the bound

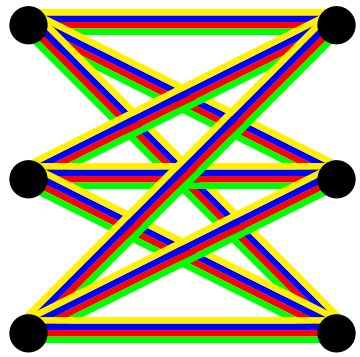
complementary slackness: asym.unique solution is Aharoni-DeVos

Mantel's theorem[4]

Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle?

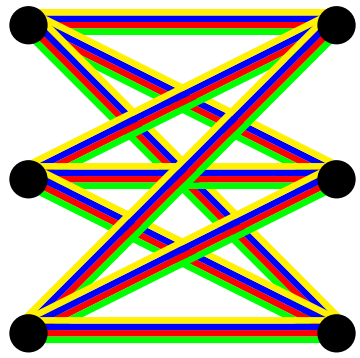
Mantel's theorem[4]

Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle? Must be $> n^2/4$



Mantel's theorem[4]

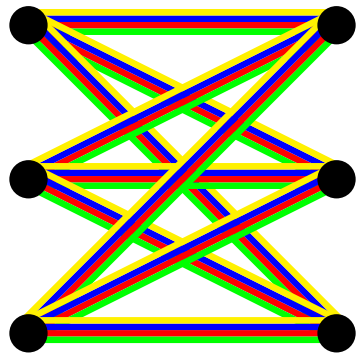
Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle? Must be $> n^2/4$



Proposition': $e(G_1) + e(G_2) + e(G_3) + e(G_4) \leq n^2$

Mantel's theorem[4]

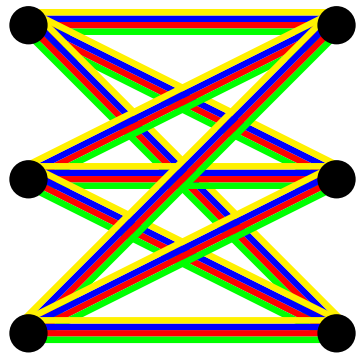
Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle? Must be $> n^2/4$



Proposition': $e(G_1) + e(G_2) + e(G_3) + e(G_4) \leq n^2$

Mantel's theorem[4]

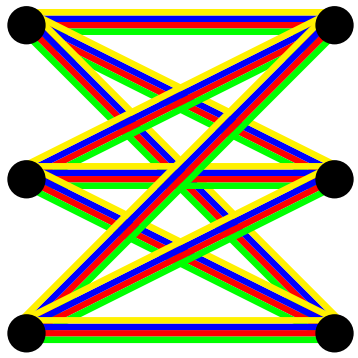
Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle? Must be $> n^2/4$



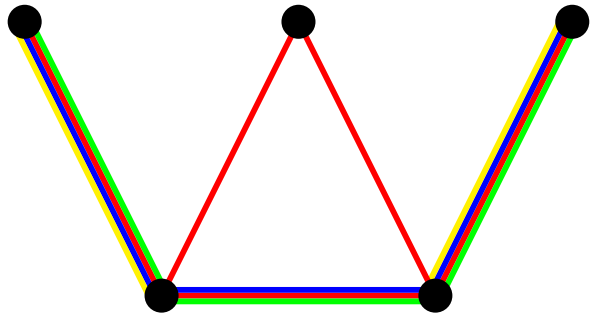
Proposition': $e(G_1) + e(G_2) + e(G_3) + e(G_4) \leq n^2$

Mantel's theorem[4]

Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle? Must be $> n^2/4$

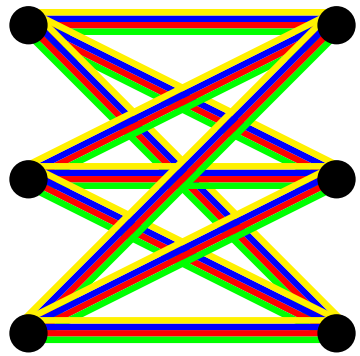


Proposition': $e(G_1) + e(G_2) + e(G_3) + e(G_4) \leq n^2$

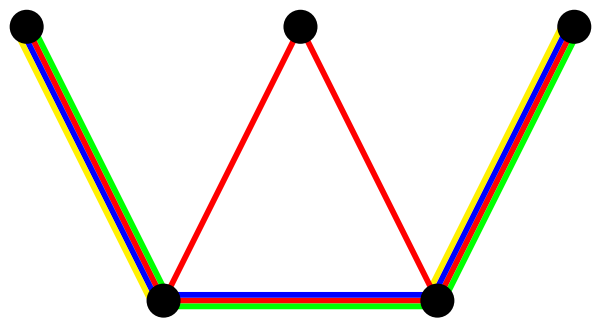


Mantel's theorem[4]

Given G_1, G_2, G_3, G_4 on $V = [n]$, what edge-bounds guarantee rainbow triangle? Must be $> n^2/4$



Proposition': $e(G_1) + e(G_2) + e(G_3) + e(G_4) \leq n^2$

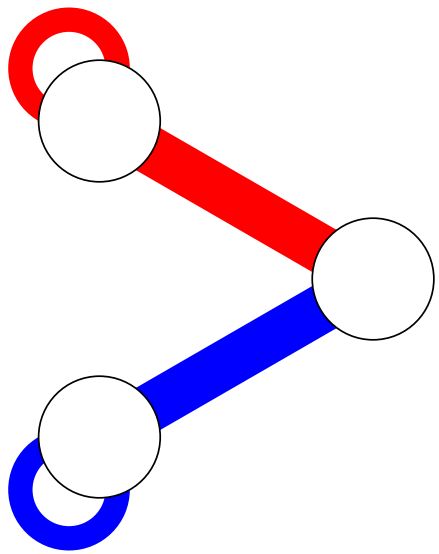


$\Longrightarrow \sum e(G_i) \leq (n-2)^2 + 4(n-2) + 4 = n^2$ □

Non-monochromatic Δ in RED/BLUE-edge-colored graphs

Non-monochromatic Δ in RED/BLUE-edge-colored graphs

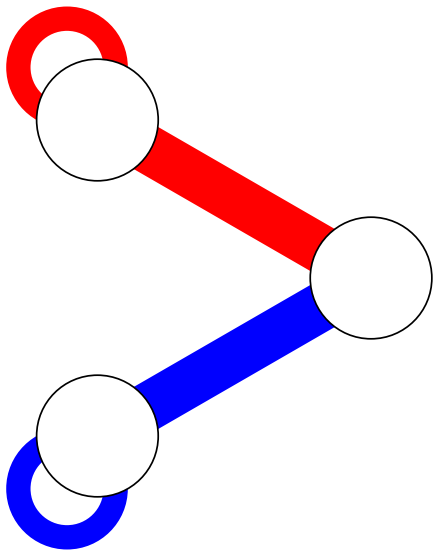
Theorem (DeVos-McDonald-Montejano 2018):
RED/BLUE-edge-colored G with RED $> \frac{1}{3} \cdot \frac{n^2}{2}$ and BLUE $> \frac{1}{3} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.

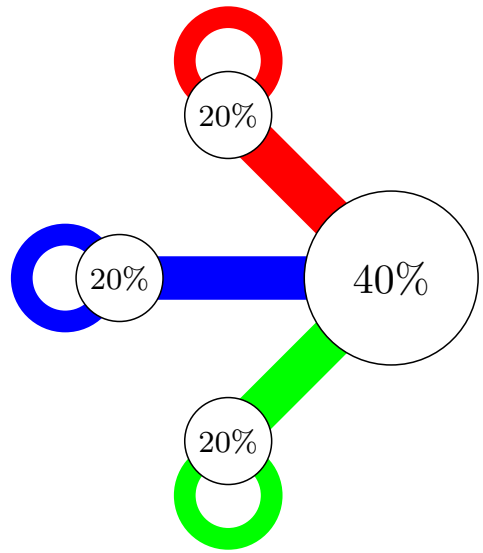


Non-monochromatic Δ in RED/BLUE-edge-colored graphs

Theorem (DeVos-McDonald-Montejano 2018):
RED/BLUE-edge-colored G with RED $> \frac{1}{3} \cdot \frac{n^2}{2}$ and BLUE $> \frac{1}{3} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.

Conjecture: k-edge-colored G with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.

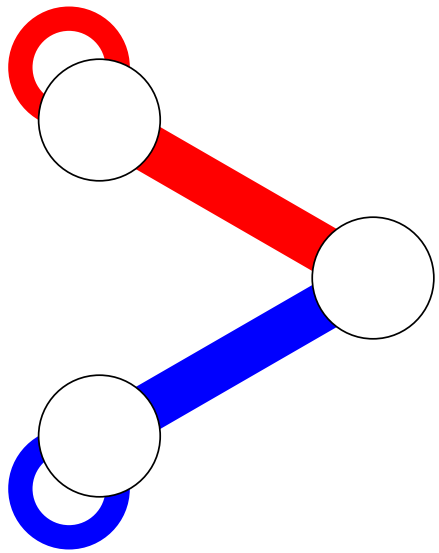


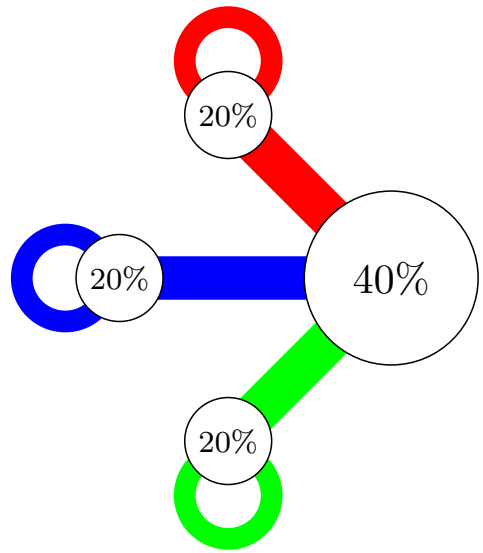


Non-monochromatic Δ in RED/BLUE-edge-colored graphs

Theorem (DeVos-McDonald-Montejano 2018):
RED/BLUE-edge-colored G with RED $> \frac{1}{3} \cdot \frac{n^2}{2}$ and BLUE $> \frac{1}{3} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.

Conjecture: k-edge-colored G with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.



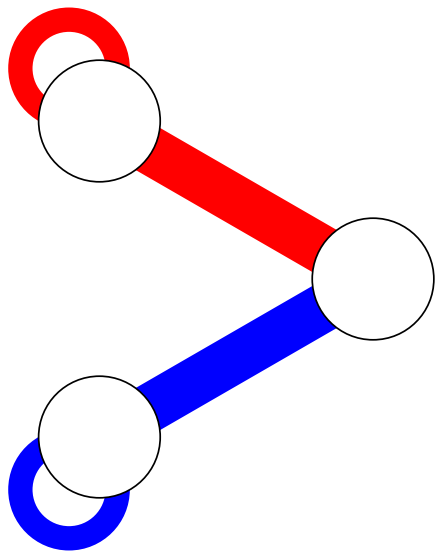


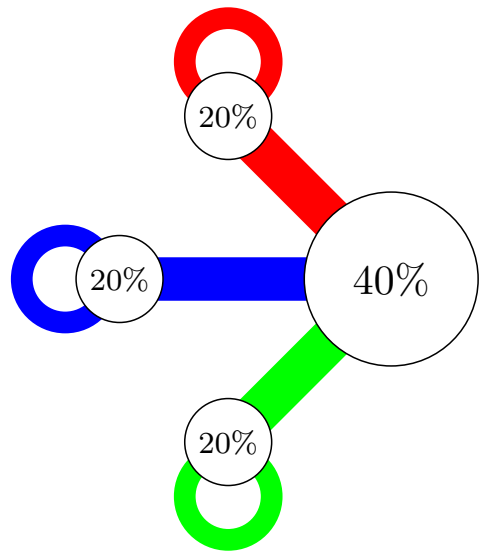
Theorem (Norin-V): For "any" given k, the conjecture is true.

Non-monochromatic Δ in RED/BLUE-edge-colored graphs

Theorem (DeVos-McDonald-Montejano 2018):
RED/BLUE-edge-colored G with RED $> \frac{1}{3} \cdot \frac{n^2}{2}$ and BLUE $> \frac{1}{3} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.

Conjecture: k-edge-colored G with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$
$\Longrightarrow$ non-monochromatic triangle in G.





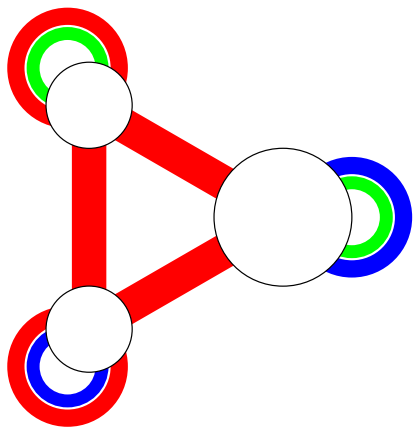
Theorem (Norin-V): For "any" given k, the conjecture is true.

WIP: a universal proof for all values k (currently $< 10^{-16}$ error)

Conclusion

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:

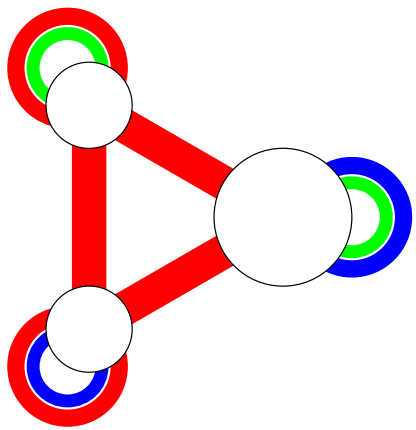
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

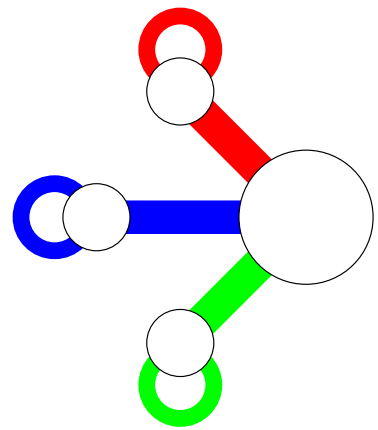


Conclusion

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

Norin-V: If G is k-edge-colored with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$,
$\Longrightarrow$ non-monochromatic triangle in G.

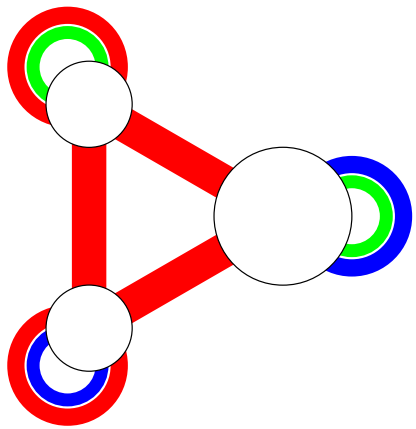


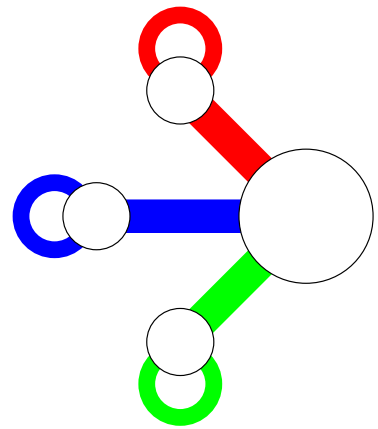


Conclusion

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

Norin-V: If G is k-edge-colored with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$, $\Longrightarrow$ non-monochromatic triangle in G.



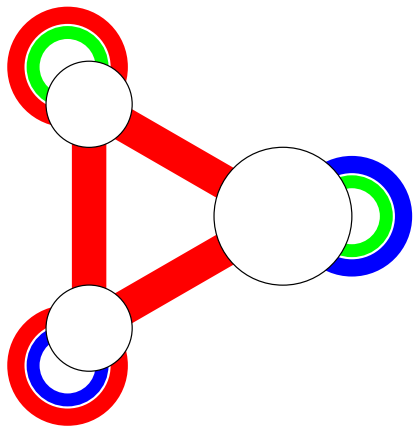


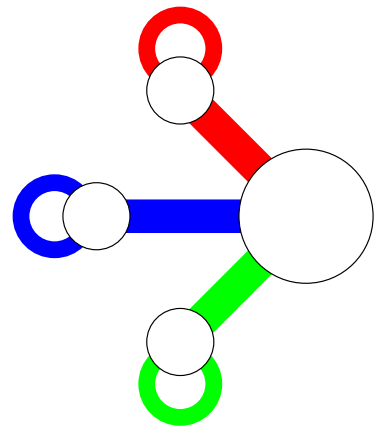
Transversal copies of H in $\left(G_1, G_2, \ldots, G_{e(H)}\right)$?

Conclusion

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

Norin-V: If G is k-edge-colored with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$, $\Longrightarrow$ non-monochromatic triangle in G.



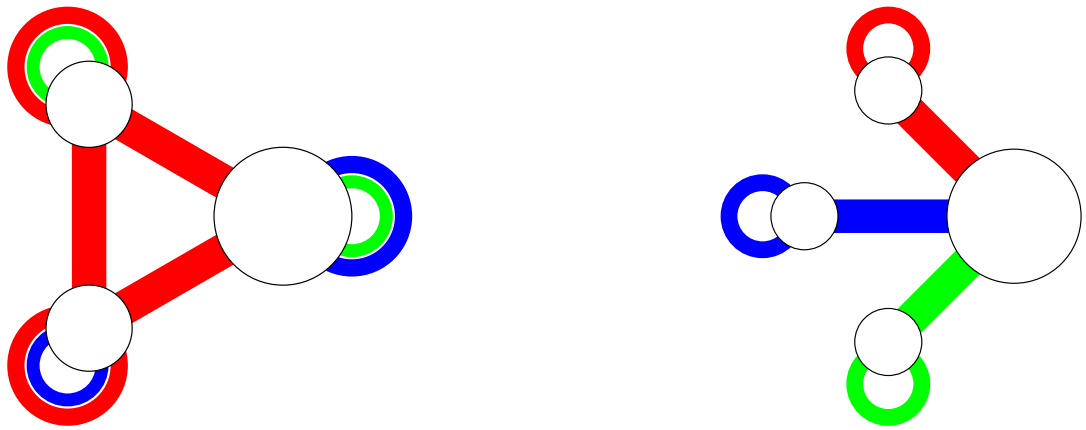


Transversal copies of H in $\left(G_1, G_2, \ldots, G_{e(H)}\right)$? #colors $> e(H)$?

Conclusion

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

Norin-V: If G is k-edge-colored with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$, $\Longrightarrow$ non-monochromatic triangle in G.



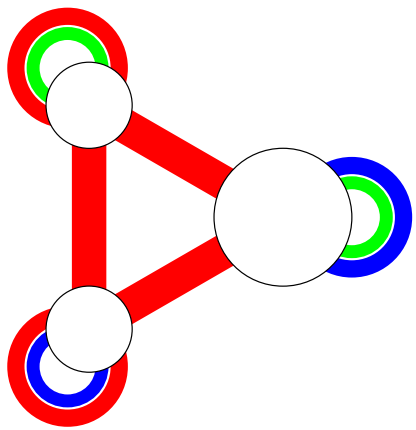
Transversal copies of H in $\left(G_1, G_2, \ldots, G_{e(H)}\right)$? #colors $> e(H)$?

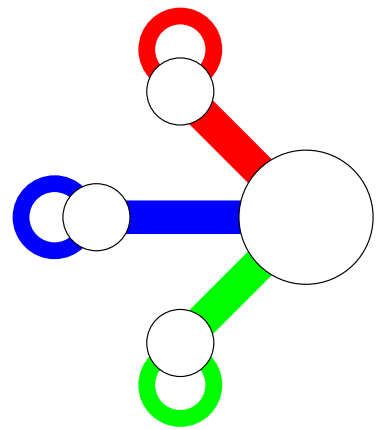
Non-mono copies of H in edge-colored G with dense color classes?

Conclusion

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

Norin-V: If G is k-edge-colored with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$, $\Longrightarrow$ non-monochromatic triangle in G.





Transversal copies of H in $\left(G_1, G_2, \ldots, G_{e(H)}\right)$? #colors $> e(H)$?

Non-mono copies of H in edge-colored G with dense color classes?

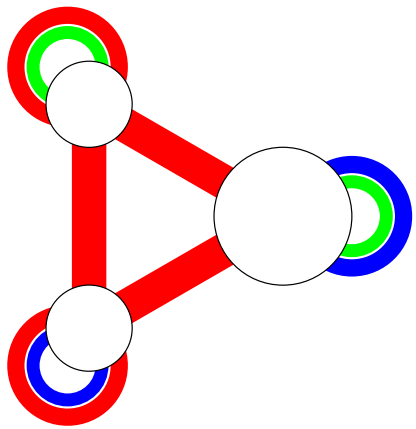
Other coloring patterns? 2-colors: conjecture by Diwan and Mubayi

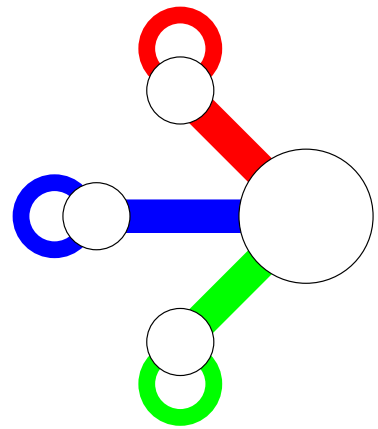
Conclusion

Thank you for your attention!

Culver-Lidický-Pfender-V / Aharoni-DeVos-Gonzales-Montejano-Šámal:
If all three graphs G_1, G_2, G_3 on $[n]$ has $> \frac{52-4\cdot\sqrt{7}}{81} \cdot \frac{n^2}{2} + O(n)$, $\Longrightarrow$ transversal triangle.

Norin-V: If G is k-edge-colored with each color class $> \frac{1}{2k-1} \cdot \frac{n^2}{2}$, $\Longrightarrow$ non-monochromatic triangle in G.





Transversal copies of H in $\left(G_1, G_2, \ldots, G_{e(H)}\right)$? #colors $> e(H)$?

Non-mono copies of H in edge-colored G with dense color classes?

Other coloring patterns? 2-colors: conjecture by Diwan and Mubayi