MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #3: Menger's theorem and network flows

Due Wednesday, November 2nd, 2016, 14:30

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1. Let $G = (V, E)$ be a simple graph and let $U \subseteq V$. We define $G \oplus_U \{v\}$ to be the graph obtained from G by adding a new vertex v, which is then joined to every vertex in U. In other words, $G \oplus_U \{v\} = (V \cup \{v\}, E \cup \{\{u, v\} : u \in U\})$.

a) Prove that if $G = (V, E)$ is a k-connected simple graph and $U \subseteq V$ has size k, then the graph $G \oplus_U \{v\}$ is k-connected as well.

b) For every integer $k \geq 1$, find a simple graph $G_k = (V_k, E_k)$ on at least $k + 1$ vertices and a vertex-subset $U \subseteq V_k$ of size k such that G_k is not k-connected, however, $G_k \oplus_U \{v\}$ is k-connected.

2. Let $G = (V, E)$ be a k-connected simple graph and $U, W \subseteq V$ two vertex-subsets, each of size k. Prove that there exist k pairwise vertex-disjoint paths $P_1, \dots P_k$ such that for every $i \in \{1, \dots, k\}$, the path P_i have one endpoint in U and the other endpoint in W.

[*Hint: Use the previous exercise.*]

3. Let $G = (V, E)$ be a 2-connected simple graph. Show that for any triple of distinct vertices $u, v, w \in V$ there is a path in G from u to v passing through w, i.e., w is one of the inner vertices of the path.

4. Let $G = (V, E)$ be a 2-connected simple graph and $v \in V$ a vertex of G. Prove that there exists a vertex $u \in V$ such that $\{u, v\} \in E$ and the graph $G - u - v$ is connected.

5. Let $G = (V, E)$ be a directed graph (digraph) and for each edge $e \in E$, let $\phi(e) \geq 0$ be a non-negative integer. Show that if for every vertex v

$$\sum_{e \in \partial^-(v)} \phi(e) = \sum_{e \in \partial^+(v)} \phi(e) \; ,$$

then there is a collection of directed cycles $C_1, ..., C_k$ (possibly with repetition) so that for every edge e of G, it holds that

$$|\{i : 1 \leq i \leq k, \; e \in E(C_i)\}| = \phi(e).$$