MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #5: Edge-colorings, Line graphs, Planar graphs

Due Wednesday, November 30th, 2016, 14:30

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1. Recall the Petersen graph depcited in Figure 1.

a) Show that the Petersen graph has no 3-edge-coloring. *(2 points)*

b) Does the Petersen graph have a Hamilton cycle? *(1 point)*

c) Find a 4-edge-coloring of the Petersen graph. *(1 point)*

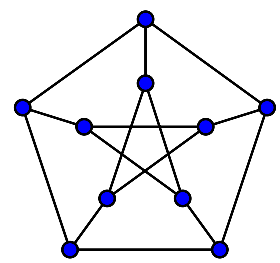

Figure 1: The Petersen graph

2. For $n \geq 2$, use the following steps to determine $\chi'(K_n)$ and construct an optimal edge-coloring.

a) For any odd integer $n \geq 3$, show that the complete graph K_n does not have an edge-coloring with $\Delta(K_n) = n - 1$ colors.

b) For any odd integer $n \geq 3$, prove that if c is an edge-coloring of K_n with n colors, then each color class of c contains $(n-1)/2$ edges. (*Note that $\chi'(K_n) \leq n$ by Vizing's Theorem.)*

c) For any even integer $n \geq 2$, show that $\chi'(K_n) = n - 1$.

d) For any integer $n \geq 2$, explicitly construct an edge-coloring of K_n with $\chi'(K_n)$ colors.

[*Hint: for n odd, put $V(K_n) = \{0, \dots, n-1\}$ and color the edge $\{i, j\}$ with $(i + j) \mod n$.*]

3. Let $G = (V, E)$ be a loopless multigraph. Recall that a *line graph* of G, which we denote by $L(G)$, is a simple graph H with the vertex set E, and two vertices e and f of H are adjacent if and only if the corresponding two edges in G are incident to the same vertex. In other words, $H = (E, F)$ where $F = \{\{e, f\} : e \cap f \neq \emptyset\}$.

a) Let $G = (V, E)$ be a loopless connected multigraph with an <u>even</u> number of edges, i.e., $|E|$ is even. Show that the graph $L(G)$ has a perfect matching.

[*Hint: use Tutte's Theorem.*]

b) Let $G = (V, E)$ be a loopless connected multigraph with an <u>odd</u> number of edges. Show that $L(G)$ has a matching of size $\frac{|E|-1}{2}$.

Please turn to the other side.

4. Let $G = (V, E)$ be a planar graph drawn in the plane. Suppose that there exists a vertex v so that v belongs to the boundary of every region. Show that

$$\alpha(G) \geq \frac{|V| - 1}{2}.$$

5. Recall a simple graph G is called *outerplanar* if it can be drawn in the plane so that every vertex is incident with the infinite region.
Let $G = (V, E)$ be a connected outerplanar graph with $|V| \geq 3$.

a) Prove that G contains two vertices of degree at most 2. *(1 point)*

b) Is it true that G necessarily contains three vertices of degree at most 2? *(1 point)*

c) Without using the 4-Color Theorem, show that $\chi(G) \leq 3$. *(2 points)*

Bonus question. *This question is worth additional* 5 *points on top of the standard* 20 *points.* Show that a graph G is outerplanar if and only if G contains no K_4-minor and no $K_{2,3}$-minor.