Instructions: The exam is 3 hours long and contains 6 questions. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. **Justify all your answers.**

Q1 Let G be the graph pictured on Figure 1.

a) Is G planar?

b) Find $\nu(G)$ and $\tau(G)$.

c) Find $\chi(G)$ and $\chi'(G)$.

Q2 Let G be the graph with weights $w : E(G) \to \mathbb{Z}_+$ pictured on Figure 2.

a) Find the min-cost spanning tree in G.

b) Find a shortest path spanning tree for the vertex A.

Q3 Let $k \geq 3$ be an integer. Let G be a bipartite graph such that

$$3 \leq \deg(v) \leq k \qquad \text{for every } v \in V(G).$$

Show that G contains a matching of size at least $\frac{3|V(G)|}{2k}$.

Q4 Let G be a loopless graph, such that G does not contain $K_{2,3}$ as a minor. Show that either $\chi(G) \leq 3$, or G contains K_4 as a subgraph.

Q5 Let G be a non-planar graph such that every subgraph of G, except for G itself, is planar. Show that $|E(G)| - |V(G)| = 3$, or $|E(G)| - |V(G)| = 5$.

Q6 Let G be a simple graph with $|V(G)| \geq 2$. Suppose that G does not contain P_4 (the path on 4 vertices) as an *induced* subgraph.

a) Prove that either G is not connected or the complement of G is not connected. (*Hint:* Use induction on $|V(G)|$. Show that, if $G \setminus v$ has at least two components and v is adjacent to a vertex in every component of $G \setminus v$, then v is adjacent to every vertex of $G \setminus v$.)

b) Deduce from a) that G is perfect. You are allowed to use the weak perfect graph theorem, but not the strong perfect graph theorem.

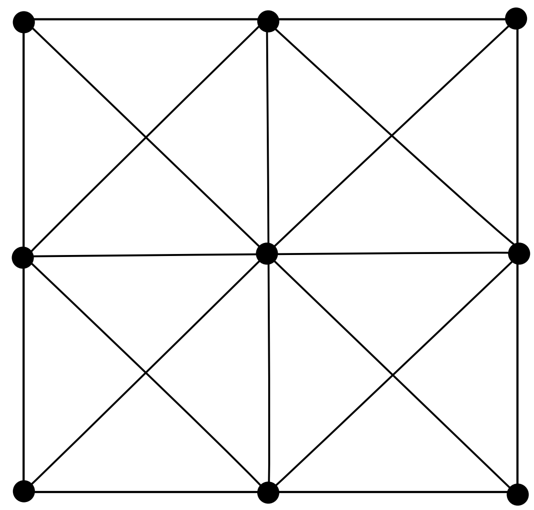

Figure 1: The graph in the question Q1.

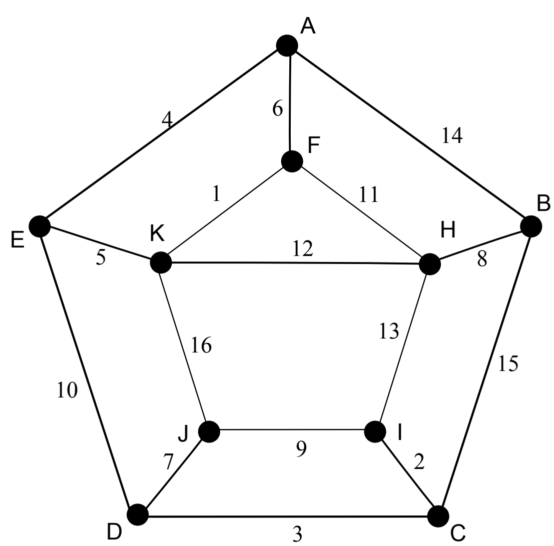

Figure 2: The graph in the question Q2.