A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that every vertex $v \in V$ is an endpoint of at most one edge in $M$. A matching is called *perfect*, if every $v \in V$ is incident to exactly one edge in $M$. For a bipartite graph $G$ with parts $A$ and $B$, we say that a matching $M$ is *$A$-covering*, if every vertex $v \in A$ is incident to exactly one edge.

For a graph $G = (V, E)$ and a set $U \subseteq V$, let

$$N_G(U) := \{x \in V | \{u, x\} \in E \text{ for some } u \in U\}$$

be *the neighborhood of $U$* in $G$.

**Theorem 1** (Hall's Theorem). *Let $G$ be a bipartite graph with parts $A$ and $B$. $G$ contains an $A$-covering matching if and only if $|N_G(A')| \geq |A'|$ for all $A' \subseteq A$.*

*Proof.* The condition $|N_G(A')| \geq |A'|$ for all $A' \subseteq A$, which we will call the *Hall's condition*, is clearly satisfied if $G$ contains an $A$-covering (because there are $|A'|$ vertices in $B$ that are incident to the edges of an $A$-covering matching, and every such vertex is contained in $N_G(A')$).

For the rest of the proof, we focus on showing that if a graph $G$ satisfies the Hall's condition, then it contains an $A$-covering matching. We proceed by induction on $|A|$. If $|A| = 1$, then the Hall's condition applied to $A' = A$ guarantees that the single vertex $v \in A$ has at least one neighbor. Let $w \in B$ be one such a neighbor. It follows that $\{v, w\}$ is an $A$-covering matching.

Suppose $|A| \geq 2$. We consider the following two exclusive options:

1. *there exists $A_0$ such that $\emptyset \subsetneq A_0 \subsetneq A$ and $|N_G(A_0)| = |A_0|$,*
2. *for every $A'$ such that $\emptyset \subsetneq A' \subsetneq A$, we have $|N_G(A')| \geq 1 + |A'|$.*

Let us start with the second option. Fix an arbitrary $v \in A$. By the Hall's condition, $v$ has at least one neighbor, say $w \in B$. Define $G' := G - v - w$. If we manage to verify the Hall's condition for $G'$, we can apply induction and obtain $(A \setminus \{v\})$-covering matching $M'$. It follows that $M := M' \cup \{v, w\}$ is then an $A$-covering matching in $G$. So it remains to check that $|N_{G'}(A')| \geq |A'|$ for every non-empty $A' \subseteq A \setminus \{v\}$. We claim this follows from that "additional +1" we have in the second option. Indeed,

$$|N_{G'}(A')| \geq |N_G(A')| - 1 \geq |A'| + 1 - 1 = |A'|,$$

where the first inequality follows from the fact that $N_{G'}(A')$ may differ from $N_G(A')$ only on the vertex $\{w\}$, and the second inequality uses the "additional +1".

It remains to resolve the first option, where there is some non-empty $A_0 \subsetneq A$ such that $|N_G(A_0)| = |A_0|$. Let $A_1 := A \setminus A_0$ and, let $G_0$ be the subgraph of $G$ induced by $V \setminus A_1$. Our aim is to apply induction on $G_0$ in order to find an $A_0$-covering matching in $G_0$. To do so, it is enough to verify whether $G_0$ satisfies the Hall's condition. But this follows easily: for every $A' \subseteq A_0$ it holds that $N_{G_0}(A') = N_G(A')$, and by the assumption $|N_G(A')| \geq |A'|$, we know that $|N_{G_0}(A')| \geq |A'|$ must hold as well. Therefore, there exists an $A_0$-covering matching $M_0$ in $G_0$, which is also a matching in $G$.

The only remaining task is to find a matching $M_1$ that will cover the vertices in $A_1$ and will be "compatible" with $M_0$, which means, the union $M_0 \cup M_1$ will be a matching in $G$. Let $G_1$ be the subgraph of $G$ induced by $V \setminus (A_0 \cup N_G(A_0))$. Our aim is to use the induction hypothesis once again, this time in order to find an $A_1$-covering matching in the graph $G_1$. If we want to use induction (again?), we need to verify the Hall's condition (again!), this time on the graph $G_1$. In other words, we need to show that $|N_{G_1}(A')| \geq |A'|$ for every $A' \subseteq A_1$. We claim that the desired inequality for a fixed $A'$ follows from the Hall's condition for $G$ applied to the set $A'' := A_0 \cup A'$. Indeed, on one hand

$$|N_G(A'')| = |N_G(A_0)| + |N_G(A') \setminus N_G(A_0)| = |A_0| + |N_{G_1}(A')|,$$

where we used that $|A_0| = |N_G(A_0)|$ and $N_G(A') \setminus N_G(A_0) = N_{G_1}(A')$. On the other hand,

$$|A''| = |A_0| + |A'|,$$

simply because the sets $A_0$ and $A'$ are disjoint. Putting these two equations together with the assumption $|N_G(A'')| \geq |A''|$ yields that

$$|N_{G_1}(A')| = |N_G(A'')| - |A_0| \geq |A''| - |A_0| = |A'| + |A_0| - |A_0| = |A'|.$$

Voila, the graph $G_1$ satisfies the Hall's condition, induction can be applied, and provides us an $A_1$-covering matching $M_1$, which by the construction of $G_1$ is completely disjoint from $M_0$. Therefore, $M_0 \cup M_1$ is a matching in $G$ that is $A$-covering. □