MATH 350: Graph Theory and Combinatorics. Fall 2016.
Assignment #2: Bipartite graphs, Matchings, Connectivity

Due Wednesday, October 19th, 2016, 14:30

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1. Recall that a graph G is called *d-regular* if every vertex of G has degree equal to d.

a) Construct a 3-regular graph that does not contain a perfect matching. You have to prove that the constructed graph does not contain a perfect matching.
Hey, I bet your graph doesn't contain a 2-factor either! A coincidence?

Solution:

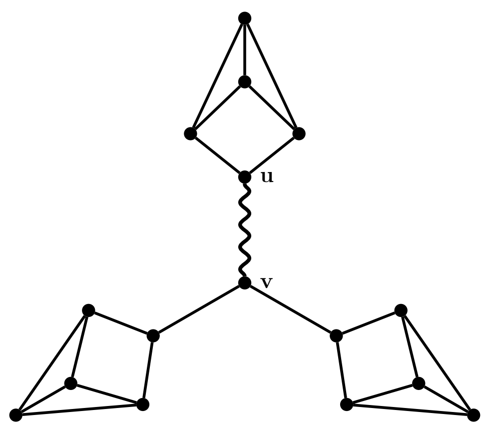

Figure 1: A 3-regular graph with no perfect matching.

Let G be the 3-regular graph depicted in Figure 1. Suppose G has a perfect matching M, and let e be the edge of M incident to the vertex v. By symmetry, we may assume $e = \{u, v\}$. Let C be any of the two connected components of $G - v$ that does not contain u. Every vertex from C is incident to one edge in M and the edges of M must lie completely inside C. But that is impossible, because $|C|$ is odd.

b) Prove the following statement: Let G be a 3-regular graph. G contains a perfect matching $\iff$ G contains a 2-factor.
...Ah, so no coincidences on this sheet!

Solution: First, suppose G has a perfect matching M. Then the spanning subgraph of G that contains the edges $E(G) \setminus M$ is a 2-factor in G.

On the other hand, if H is a 2-factor in G, then every vertex $v \in V(G)$ is incident to exactly one edge from the set $M := E(G) \setminus E(H)$. Therefore, M is a perfect matching in G.

2. Prove that every graph $G = (V, E)$ contains a subgraph H that is bipartite and has at least $|E|/2$ edges.

Solution: Let H be a bipartite subgraph of G that has maximum number of edges, and let $A \subseteq V$ and $B \subseteq V$ be the parts of the bipartition. Clearly, H contains every edge of G that has one endpoint in A (and hence also one endpoint in B). It is enough to show that $\deg_H(v) \geq \frac{1}{2}\deg_G(v)$ for every $v \in V$.
Suppose for contradiction there exists a vertex $v \in V$ such that $\deg_H(v) < \frac{1}{2}\deg_G(v)$. Without loss of generality, $v \in A$. But then the vertex v has in G more neighbors inside A than inside B. Therefore, the bipartite graph H' between the parts $A' := A \setminus \{v\}$ and $B' := B \cup \{v\}$ contains more edges than H, a contradiction.

3. Let G be a connected graph. We say that $F \subseteq E(G)$ is *even-degree*, if every vertex of G is incident with an even number of edges in F. Let T be an arbitrary spanning tree of G. Prove that there exists an even-degree set $F_T \subseteq E(G)$ such that $F_T \cup E(T) = E(G)$.

Solution: We claim that if F_1 and F_2 are both even-degree then so is $F_1 \triangle F_2 := (F_1 - F_2) \cup (F_2 - F_1)$. Indeed, if E_1 and E_2 are the sets of edges in F_1 and F_2, respectively, incident to the vertex v, then $|E_1 \triangle E_2| = |E_1| + |E_2| - 2|E_1 \cap E_2|$, which is even if $|E_1|$ and $|E_2|$ are even.
For $e \in E(G) \setminus E(T)$, let $F(e)$ be the edge set of the cycle formed by e and the unique path between the endpoints of e in T. Clearly, $F(e)$ is even-degree. Let

$$F_T := F(e_1) \triangle F(e_2) \ldots \triangle F(e_k),$$

where $E(G) \setminus E(T) = \{e_1, e_2, \ldots, e_k\}$. Then F is an even-degree set, by the claim above, and $F \cup E(T) = E(G)$, as $e_i \in F(e_i)$ and $e_i \notin F(e_j)$ for $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$.

4. Let G be a 3-regular graph. Show that the edge connectivity $\kappa_e(G)$ is equal to the vertex connectivity $\kappa_v(G)$.

Solution: First observe that for an arbitrary graph G, it holds that $\kappa_v(G) \leq \kappa_e(G)$. Indeed, any internally k vertex-disjoint paths between any two vertices form also k edge-disjoint paths between the two vertices. So it is enough to show $\kappa_v(G) \geq \kappa_e(G)$. Also, since the graph G is 3-regular, $\kappa_e(G) \leq 3$. Therefore, if $\kappa_v(G) = 3$, then there is nothing to prove. On the other hand, if $\kappa_v(G) = 0$, then G is disconnected and so $\kappa_e(G) = 0$ as well. It remains to analyze $\kappa_v(G) \in \{1, 2\}$.
If $\kappa_v(G) = 1$, then by Menger's theorem there exists a cut-vertex $v \in V(G)$ such that $G - v$ is disconnected. Let C_1 and C_2 be any two different connected components of $G - v$. Since the degree of v is only three, there must be $i \in \{1, 2\}$ such that v is adjacent to only one vertex $w \in V(C_i)$. But then $\{v, w\}$ is a cut-edge in G and hence $\kappa_e(G) \leq 1$.
Finally, consider $\kappa_v(G) = 2$. Again, by Menger's theorem there exists two vertices u and v so that $G - u - v$ is disconnected. And again, let C_1 and C_2 be any two different connected components of $G - u - v$. Clearly, both u and v have at least one neighbor in C_1 and at least one neighbor in C_2 (as otherwise $\kappa_v(G) = 1$). We now consider the following two cases:

- If $\{u, v\} \in E(G)$, then since $\deg_G(u) = 3$, it must be that u has exactly one neighbor in C_1, call it u_1, and exactly one neighbor in C_2, call it u_2. Analogously, v has exactly one neighbor in C_1, say v_1, and exactly one neighbor in C_2, call it v_2. But then both $\{\{u, u_1\}, \{u, u_2\}\}$ and $\{\{v, v_1\}, \{v, v_2\}\}$ are edge cuts in G, and hence $\kappa_e(G) \leq 2$.
- If $\{u, v\} \notin E(G)$, then there exists $i \in \{1, 2\}$ such that u has exactly one neighbor in C_i (this time, we do not claim anything about the number of its neighbors in C_{3-i}). Let u_i be the neighbor of u in C_i. There also exists $j \in \{1, 2\}$ such that v has exactly one neighbor in C_j; we denote this neighbor by v_j. It follows that $\{\{u, u_i\}, \{v, v_j\}\}$ is an edge cut in G of size 2.

5. Let G be an n-vertex bipartite graph such that every degree of G is between 10 and 20. Show that G contains a matching of size at least $n/3$.

Solution: By Kőnig's theorem, it is enough to show that the size of any vertex cover of G is at least $n/3$. Let X be a vertex cover of G. Every vertex $v \in X$ is incident to $\deg_G(v)$ edges, but every edge of G is incident to at least one $v \in X$, hence

$$|E(G)| \leq \sum_{v \in X} \deg_G(v).$$

On the other hand,

$$2|E(G)| = \sum_{v \in V(G)} \deg_G(v) = \left(\sum_{v \in X} \deg_G(v) \right) + \left(\sum_{v \in V(G) \setminus X} \deg_G(v) \right).$$

Therefore, combining the two estimates on $|E(G)|$ together and using the fact that each degree is between 10 and 20 yield

$$20 \cdot |X| \geq \sum_{v \in X} \deg_G(v) \geq \sum_{v \in V(G) \setminus X} \deg_G(v) \geq 10 \cdot (n - |X|).$$

But this immediately implies $|X| \geq n/3$.