MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #2: Trees

Due Thursday, September 28st, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.
An automorphism of a graph $G = (V, E)$ is a permutation π on V such that $uw \in E \iff \pi(u)\pi(w) \in E$ for all $u, w \in V$. G is called asymmetric if $|V(G)| \geq 2$ and the only automorphism G has is the identity.

a) Construct an asymmetric simple graph (try to find as small asymmetric graph as you can). *(0 points)*

b) Construct an asymmetric tree (try to find as small asymmetric tree as you can). *(0 points)*

c) Let G be an n-vertex graph and let $\mathcal{A}_G \subseteq S_n$ be the set of all automorphisms of G. Prove that $\mathcal{A}_G$ together with the function composition operation $\circ$ form a group, i.e., *(0 points)*

- $\pi \in \mathcal{A}_G \iff \pi^{-1} \in \mathcal{A}_G$, and
- if $\pi, \sigma \in \mathcal{A}_G$ then $\pi \circ \sigma \in \mathcal{A}_G$.

d) Prove that every loopless graph $G = (V, E)$ with $|E| \geq 1$ contains $v \in V$ such that *(0 points)*

$$\mathrm{comp}(G - v) = \mathrm{comp}(G).$$

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1. Prove that a graph $G = (V, E)$ is a tree if and only if G has no cycles and $|V| = |E| + 1$. *(3 points)*

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2. Let T be a tree. Recall that a leaf of T is a vertex of degree one.

a) Prove that T has exactly two leaves if and only if T is a path on at least two vertices. *(1 point)*

b) Prove that if T contains a vertex of degree k, then T contains at least k leaves. *(1 point)*

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3. Let K_n^- be the graph with the vertex-set $\{1, 2, \ldots, n\}$ and the edge-set $\binom{[n]}{2} - \{1, 2\}$. Find a simple closed-form formula for the number of spanning trees of K_n^-, and prove it is correct. *(5 points)*

[*Hint: Remember you know Cayley's formula for the number of spanning trees of K_n.*]

($\star$) **This is a challenge of the week question, do not submit your solution.**

Recall that for a tree $T = (V, E)$ and a vertex $v \in V$, the pair (T, v) is called the rooted tree (with the root v) and denoted by T_v. Two rooted trees T_v and $T'_{v'}$ are isomorphic if there exists a bijective map $f : V(T) \to V(T')$ such that $f(v) = v'$ and $uw \in E(T) \iff f(u)f(w) \in E(T')$ for every $u, w \in V(T)$.

a) Let $\mathcal{U}'_n$ be the set of all n-vertex rooted unlabelled trees up to isomorphism. Construct a "natural" injective map from $\mathcal{U}'_n \to \{0, 1\}^{2n}$. *(0 points)*

[*Hint: map the single-vertex rooted tree to the sequence $(0, 1)$; for a rooted tree with at least 2 vertices, define its image recursively by removing the root v and considering the images of the connected components of $T - v$.*]

b) Deduce that there are at most 4^n rooted unlabelled trees with n vertices, and hence also at most 4^n unlabelled trees with n vertices. *(0 points)*