MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #3: Components, Minimum Spanning Trees, Shortest Paths

Due Thursday, October 5th, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Design an algorithm that for a given weighted graph G finds its *maximum spanning tree*, i.e., among all the spanning trees of G it finds the one which has the largest sum of the edge-weights. *(0 points)*

b) Construct a weighted graph G (with some edges having negative weights) that has two vertices $s, t \in V(G)$ such that the Dijkstra's algorithm will fail to find the shortest path between s and t. *(0 points)*

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1. Determine what is the maximum number of edges of an n-vertex graph G with k connected components (i.e., $\text{comp}(G) = k$). *(2 points)*

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2. Let V be a finite set, and let $d : V \times V \to \mathbb{N} \cup \{0\}$ be any non-negative integral function that satisfies:

1) $d(u, w) = 0 \iff u = w$,
2) $d(u, w) = d(w, u)$ for every $u, w \in V$,
3) $d(u, w) \leq d(u, v) + d(v, w)$ for every $u, v, w \in V$, and
4) if $d(u, w) > 1$, then there exists $v \in V \setminus \{u, w\}$ such that $d(u, w) = d(u, v) + d(v, w)$.

Prove that there exists a simple graph $G = (V, E)$ such that $\text{dist}_G = d$. *(4 points)*

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3. Let $G = (V, E)$ be a simple graph. A set $A \subseteq V$ is called *an independent set in G* if the induced subgraph $G[A]$ contains no edge, i.e., if every edge $e \in E$ is incident to at most one vertex in A. Define $\alpha(G)$ to be the maximum cardinality of an independent set in G.
Suppose $G = (V, E)$ is a simple graph. Prove that there is an integer $\ell \leq \alpha(G)$ and a collection of paths $P_1, P_2, \ldots, P_\ell$ such that for every vertex $v \in V$ there exists exactly one path P_i, where $i \in \{1, \ldots, \ell\}$, with $v \in V(P_i)$. *(4 points)*

[*Hint: From all the collections of paths $P_1, P_2, \ldots, P_\ell$ in G that has the desired property that each vertex is in exactly one path (prove first there is always at least one such collection, BTW!), take the collection that minimizes the value of ℓ. What can you say if $\ell > \alpha(G)$?*]

($\star$) **This is a challenge of the week question, do not submit your solution.**

This is a strengthening of the previous exercise. For an oriented graph $D = (V, E, o)$, we define an *oriented path* in D to be a sequence $u_0, e_1, u_1, \ldots, e_k, u_k$, where $\{u_0, u_1, \ldots, u_k\} \subseteq V$, $\{e_1, \ldots, e_k\} \subseteq E$, and the arc (oriented edge) e_i is oriented from the vertex u_{i-1} to the vertex u_i for all $i \in \{1, 2, \ldots, k\}$. The definition of an independent set directly translates to oriented graphs, i.e., it is a subset of V that contains no arc.

Suppose D is an oriented graph. Prove that there is an integer $\ell \leq \alpha(G)$ and a collection of oriented paths $P_1, P_2, \ldots, P_\ell$ such that for every vertex $v \in V$, there exists exactly one oriented path P_i, where $i \in \{1, \ldots, \ell\}$, with $v \in V(P_i)$. *(0 points)*