MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #4: Eulerian Tours, Bipartite graphs

Due Thursday, October 12th, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Let $G = (V, E)$ be a multigraph that has all its degrees being even. Prove that there exists a collection of cycles $C_1, C_2, \ldots, C_\ell$ in G such that for every edge $e \in E$ there exists *exactly one* cycle C_i, where $i \in \{1, \ldots, \ell\}$, with $e \in E(C_i)$. *(0 points)*

[*Hint: Use induction on* $|E(G)|$.]

b) Let $G = (V, E)$ be a connected graph and $v \in V$ a vertex of G. Prove that G is bipartite if and only if $\mathrm{dist}_G(u, v) \neq \mathrm{dist}_G(w, v)$ for all $\{u, w\} \in E$. *(0 points)*

c) Prove that every n-vertex bipartite graph G has at most $\lfloor n^2/4 \rfloor$ edges, and characterize all the bipartite graphs G with exactly this many edges for every $n \in \mathbb{N}$. *(0 points)*

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1. Let G be a connected multigraph. We say that $F \subseteq E(G)$ is *even-degree*, if every vertex of G is incident with an even number of edges in F. Let T be an arbitrary spanning tree of G. Prove that there exists an even-degree set $F_T \subseteq E(G)$ such that $F_T \cup E(T) = E(G)$. *(3 points)*

[*Hint: First, prove that if two sets F and F' are both even-degree, then so is the set*

$$F \triangle F' := (F \setminus F') \cup (F' \setminus F).]$$

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2. Let $G = (V, E)$ be a connected multigraph that has exactly 2ℓ vertices of odd degree. Prove that there exists a collection of trails $T_1, T_2, \ldots T_\ell$ in G such that every edge $e \in E$ is in exactly one trail T_i, where $i \in \{1, \ldots, \ell\}$. *(4 points)*

In other words, we can draw G on a paper using only ℓ lines such that for every line, we do not lift the pencil, and each edge of the graph was drawn exactly once.

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3. Prove that every loopless graph $G = (V, E)$ contains a subgraph H that is bipartite and $\deg_H(v) \geq \lceil \deg_G(v)/2 \rceil$ for every $v \in V$. In particular, H has at least $\lceil |E|/2 \rceil$ edges. *(3 points)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

Prove that every loopless graph $G = (V, E)$ with $|V| = 2n$ contains a subgraph H that is bipartite, both parts of H have size n, and the number of edges of H is at least $\lceil |E|/2 \rceil$. *(0 points)*

Compare this result with the previous exercise!