MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #5: Matchings in bipartite graphs

Due Thursday, October 19th, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Construct a 3-regular graph with no perfect matching. *(0 points)*

b) Let G be a 3-regular graph. Prove that G has a perfect matching $\iff$ G has a 2-factor. *(0 points)*

c) Let $G = (V, E)$ be a simple bipartite graph with parts A and B such that $|A| = |B| = n$. If the minimum degree $\min_{v \in V} \deg_G(v) \geq n/2$, then G contains a perfect matching. Also show that the degree condition $n/2$ cannot be improved to $n/2 - 1$. *(0 points)*

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1. Imagine that the McGill Extended Squad of Silliness (MESS) posts a new regulation regarding the student clubs at McGill. Specifically, the new rules are:

1) No student can be a member of more than 50 different clubs,
2) Every club must have its president, and the president must be a member of the club,
3) No student can serve as the president for more than 5 different clubs at a time.

However, implementing the rules turn the student club situation at McGill into a complete mess (thank you, MESS!), and it is now not possible to follow the rules and assign presidents for all the clubs. In order to fix this, MESS intends to introduce an additional rule stating:

4) Every club must have at least K members (all the clubs with fewer members will be closed).

This is the moment when MESS wants to hire you for a consultation: what is the minimum value of K so that it is always guaranteed that there will exists an assignment of the presidents to the clubs fulfilling the requirements (1)-(4)? MESS offers to pay you 3 points for MATH 350 if you can give them the best value of K, and also prove that with $K - 1$ it can happen that there is no such an assignment. *(3 points)*

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2. An $n \times n$ *Latin square* is a table with n rows and n columns, where each cell contains one number between 1 and n in such a way that in each row every number appears exactly once, and also in every column each number appears exactly once (similarly as in Sudoku). See examples of a 3×3 and a 4×4 Latin squares.

1	2	3
2	3	1
3	1	2

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

Analogously, for $m \leq n$, an $m \times n$ *Latin rectangle* is a table with m rows and n columns where each cell contains one number between 1 and n in such a way that in each row every number appears exactly once, and in every column each number appears at most once.
Prove that for any $m \times n$ Latin rectangle there exists an $n \times n$ Latin square so that the first m lines of the Latin square are equal to the lines of the rectangle. *(3 points)*

Please turn to the other side.

3. For a graph $G = (V, E)$ and $S \subseteq V$, recall that $N(S) := \bigcup_{u \in S} N(u) = \{v \in V \mid \exists u \in S \wedge uv \in E\}$.

a) Let $G = (V, E)$ be a bipartite graph. Prove that G has a perfect matching if and only if $|N(S)| \geq |S|$ for every $S \subseteq V$. *(3 points)*

b) Construct a connected non-bipartite graph $G = (V, E)$ with $|V|$ even that has no perfect matching but yet it satisfies that $|N(S)| \geq |S|$ for every $S \subseteq V$. *(1 point)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

Let $G = (V, E)$ be a simple bipartite graph with parts A and B. Suppose that $\deg_G(a) \geq \deg_G(b)$ for every $ab \in E$ with $a \in A$ and $b \in B$. Prove that then G contains an A-covering matching. *(0 points)*