MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #7: Ramsey Theory

Due Thursday, November 2nd, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.

a) Construct a red/blue coloring of $E(K_8)$ such that the coloring contains neither red K_3 nor blue K_4.

b) Prove that $R(3,4) = 9$. *(0 points)*

c) Prove that $R(4,4) \leq 18$. *(0 points)*

[*Note that there exists a coloring of $E(K_{17})$ coming from number theory that has no monochromatic K_4.*]

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1. For given integers k, ℓ and m, recall that $R(k,\ell,m)$ is the smallest integer N such that any red/blue/green coloring of $E(K_N)$ contains at least one of the following subgraphs: a red copy of K_k, a blue copy of K_ℓ, or a green copy of K_m. Prove that

$$R(k,\ell,m) \leq \frac{(k+\ell+m-3)!}{(k-1)!(\ell-1)!(m-1)!} \,.$$

(3 points)

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2. Let $R_k(3) := R(\overbrace{3,3,\ldots,3}^{k})$ is the minimum integer n such that any k-coloring of $E(K_n)$ contains a monchromatic K_3. Prove that $R_k(3) \leq 3k!$ for any integer $k \geq 1$. *(3 points)*

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3. A *torunament* is an oriented graph where every two vertices u and v are joined by either an oriented edge from u to v ($u \to v$), or from v to u. More formally, it is a triple (V,E,o) where V is a set of vertices, $E = \binom{V}{2}$ a set of edges, and $o : E \to V$ such that $o(\{u,v\}) \in \{u,v\}$ a function determining the orientation by letting $o(e)$ to be the source of the edge $e \in E$.
An *oriented cycle* in a tournament $T = (V,E,o)$ is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ where $v_i \in V$, $e_i \in E$, $v_1 = v_{k+1}$, $e_i = \{v_i, v_{i+1}\}$ and $o(e_i) = v_i$ for all $i \in \{1,2,\ldots,k\}$. A vertex-subset U of V is called *acyclic set* in T if the subtournament of T induced by U contains no oriented cycle.

Let T be an n-vertex tournament. Prove that T contains an acyclic set of size $\lfloor \log_2(n) \rfloor + 1$. *(4 points)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

Let $\mathcal{G}_n$ be the set of all the n-vertex simple graphs that contain no triangle.

a) For all but finite number of values of $n \in \mathbb{N}$, prove there exists $G \in \mathcal{G}_n$ with $\alpha(G) \leq n^{0.99}$. Can you do even better than 0.99 in the exponent? Say $\alpha(G) = O\left(n^{2/3}\right)$? Or even $\alpha(G) = O\left(n^{0.51}\right)$? *(0 points)*

[*The current record in this direction is: for every $\varepsilon > 0$ there is n_0 such that for all $n \geq n_0$, there exists $G \in \mathcal{G}_n$ with $\alpha(G) < (1+\varepsilon) \cdot \sqrt{2n\log(n)}$.*]

b) Prove there exists a constant $c > 0$ such that every $G \in \mathcal{G}_n$ has $\alpha(G) > c \cdot \sqrt{n\log(n)}$. *(0 points)*

[*The current record in this direction is: $\alpha(G) > \sqrt{\frac{1}{2} \cdot n\log(n)}$ for all $G \in \mathcal{G}_n$.*]