MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #9: Proper vertex-colorings of graphs

Due Tuesday, November 21st, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution. However, if you have any trouble with solving it, get in touch with me for a hint.
Recall that for a simple graph G, the chromatic number $\chi(G)$ is the minimum number of colors needed to color the vertices of G so that for every edge e the ends of e receive two different colors.
Let $G = (V, E)$ be a simple graph with $\chi(G) = k$.

a) Prove that $|E| \geq \binom{k}{2}$. *(0 points)*

b) Prove that $|E| \leq \frac{k-1}{2k}|V|^2$. *(0 points)*

c) Prove that G has a subgraph $H \subseteq G$ such that $\deg_H(v) \geq k-1$ for every $v \in V(H)$. *(0 points)*

d) Prove that there exists an ordering o of the vertex-set V such that the greedy coloring algorithm applied on G with the ordering o will construct a proper vertex-coloring of G using k colors. *(0 points)*

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1. Let $G = (V, E)$ be a simple graph and $\overline{G} = \left(V, \binom{V}{2} - E\right)$ its complement. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq |V|$. *(3 points)*

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2. Let G be a simple graph such that for any two odd cycles C_1 and C_2 in G it holds that $V(C_1) \cap V(C_2) \neq \emptyset$. Prove that $\chi(G) \leq 5$. *(3 points)*

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3. Prove that there exists a constant $C > 0$ such that the following is true: If G is an n-vertex graph with no subgraph isomorphic to K_4, then $\chi(G) \leq C \cdot n^{2/3}$. *(4 points)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

For every $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ construct a graph $G_{k,\ell}$ such that $\chi(G) > k$ and G contains no cycle of length at most ℓ. *(0 points)*