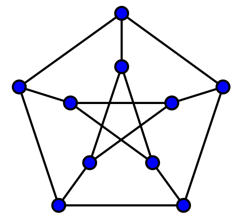
MATH 350: Graph Theory and Combinatorics. Fall 2017.
Assignment #10: Proper edge-colorings of graphs

Due Thursday, November 23st, 8:30AM *Write your answers clearly. Justify all your answers.*

This is a warm-up question, do not submit your solution.



a) The Petersen graph P is the 10-vertex 3-regular graph depcited above. Prove that P has no 3-edge-coloring, and find a 4-edge-coloring of P. In other words, prove that $\chi'(P) = 4$. *(0 points)*

b) Let $G = (V, E)$ be a loopless multigraph. Prove that $\chi'(G) \geq \frac{|E|}{\lfloor n/2 \rfloor}$. *(0 points)*

c) Let $G = (V, E)$ be a loopless multigraph and let $H \subseteq G$ be a subgraph of G with k vertices such that $k \geq 3$ and odd. Prove that

$$\chi'(G) \geq \left\lceil \frac{2|E(H)|}{k-1} \right\rceil .$$ *(0 points)*

1a) Prove that if G is a 3-regular simple graph that contains a Hamilton cycle, then $\chi'(G) = 3$. *(2 points)*

1b) Construct a simple 3-regular graph with $\chi'(G) = 3$ that contains no Hamilton cycle. *(1 point)*

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2. For $n \geq 2$, use the following steps to determine $\chi'(K_n)$ and construct its optimal edge-coloring:

a) For every odd integer $n \geq 3$, observe that K_n does not have an edge-coloring with $n-1$ colors. *(1 point)*

b) For every odd integer $n \geq 3$, prove that if c is an edge-coloring of K_n with n colors, then each color class of c contains $(n-1)/2$ edges. (Note that $\chi'(K_n) = n$ follows from Vizing's Theorem) *(1 point)*

c) For every even integer $n \geq 2$, use (b) to show that $\chi'(K_n) = n - 1$. *(1 point)*

d) For every integer $n \geq 2$, explicitly construct an edge-coloring of K_n with $\chi'(K_n)$ colors. *(1 point)*

[*Hint for (d): if n is odd, put $V(K_n) = \{0, \ldots, n-1\}$ and color the edge $\{i, j\}$ with $(i + j) \mod n$.*]

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3. Let $G = (V, E)$ be a loopless multigraph. Recall the *line graph* of G, which we denote by $L(G)$, is a simple graph with the vertex set being E, and $e \in E$ is adjacent to $f \in E$ in $L(G)$ if and only if the edges e and f of G have an endpoint in common. Equivalently, $L(G) = (E, F)$ where $F = \{\{e, f\} : e \cap f \neq \emptyset\}$.

a) Let $G = (V, E)$ be a loopless connected multigraph with an even number of edges. Prove that the line graph $L(G)$ has a perfect matching. *(2 points)*

[*Hint for (a): use Tutte's Theorem.*]

b) Let $G = (V, E)$ be a loopless connected multigraph with an odd number of edges. Prove that $L(G)$ has a matching of size $\frac{|E|-1}{2}$. *(1 point)*

($\star$) **This is a challenge of the week question, do not submit your solution.**

Construct an infinite sequence of 3-regular simple graphs G_n, where $n \in \mathbb{N}$, such that every G_n is 3-connected and has $\chi'(G_n) = 4$. *(0 points)*