Instructions: The exam is 3 hours long and contains 6 questions. The total number of points is 100. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. **Justify all your answers!**

Q1 Let G be the graph depicted in Figure 1.

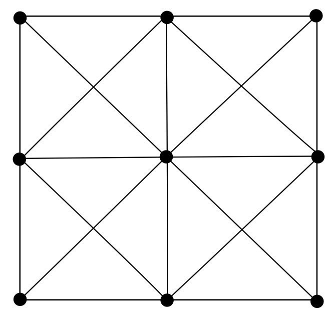
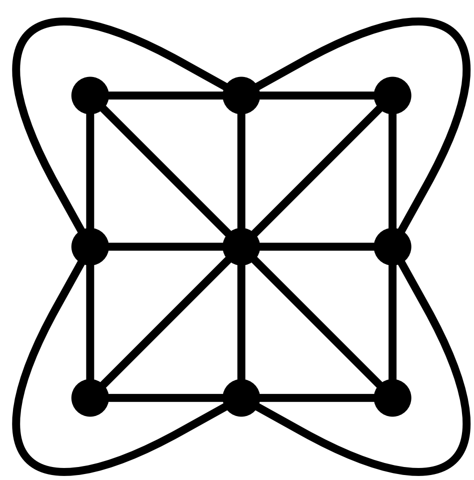


Figure 1: The graph in the question Q1.

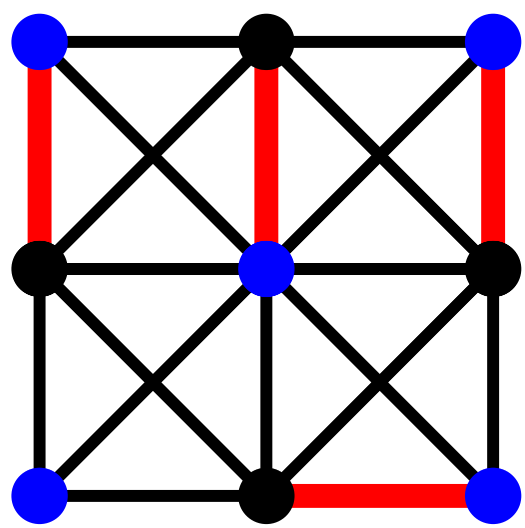
a) Is G planar? *(4 points)*

Solution: Yes. See the following drawing of G.



b) Find $\nu(G)$ and $\tau(G)$. *(4 points)*

Solution: G has 9 vertices, so $\nu(G) \leq \lfloor \frac{9}{2} \rfloor = 4$. On the other hand, the red edges on the picture below shows a matching of size 4 in G. We already know that $\tau(G) \geq \nu(G) = 4$. The blue vertices on the picture below shows a vertex cover of size 5; in the rest we show that $\tau(G) > 4$. Suppose there is a vertex cover X in G of size 4. Then X must contain the middle vertex v of G, otherwise X needs to contain all the other eight vertices. However, that implies that the set $X - v$ is a vertex cover of size 3 in the graph $G - v \supset C_8$. But that is a contradiction (for example recall that $\alpha(C_8) = 4$ and $\alpha(H) + \tau(H) = |V(H)|$ for any graph H).



c) Find $\chi(G)$. *(4 points)*

Solution: Since G contains K_4 as a subgraph $\implies \chi(G) \geq 4$. See on the left picture below a 4-coloring of G which shows $\chi(G) \leq 4$.

d) Find $\chi'(G)$. *(4 points)*

Solution: The middle vetex has degree 8 $\implies \chi'(G) \geq 8$. See on the right picture below an 8-edge-coloring of G which shows $\chi'(G) \leq 8$.

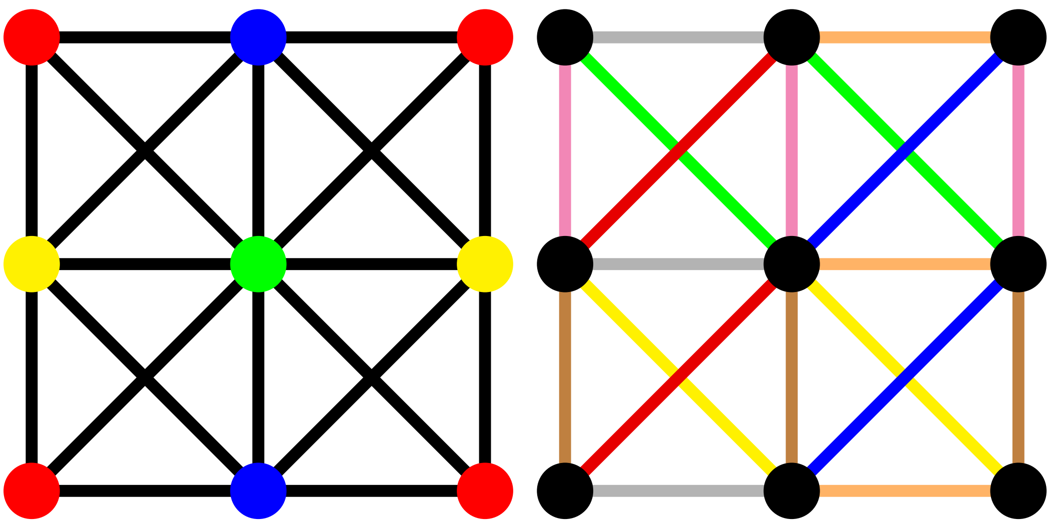


Figure 2: A 4-coloring of G from Q1 with red,blue,green and yellow. An 8-edge-coloring of G from Q1 with red,blue,green,yellow,orange,gray,purple and brown.

Q2 Let $\overrightarrow{G} = (V, E)$ be the oriented graph with the two specific vertices s and t and with the capicities $c : E \to \mathbb{Z}_+$ depicted in Figure 3.

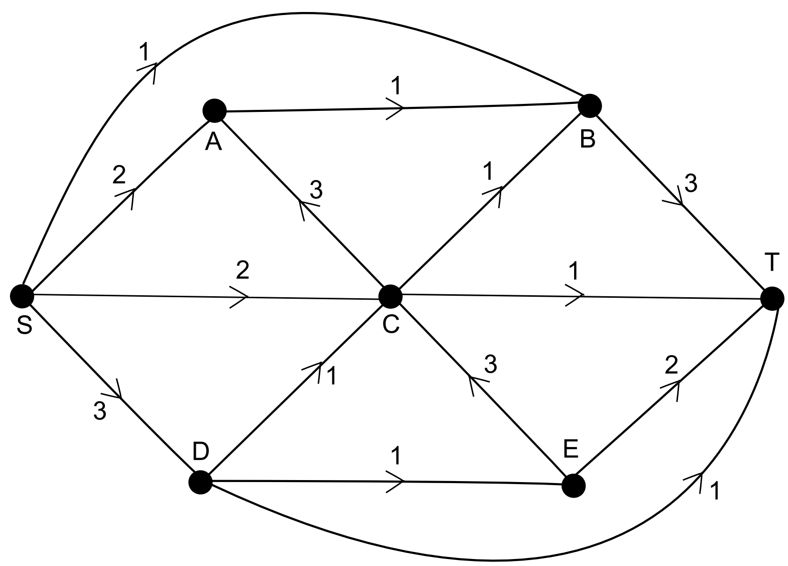
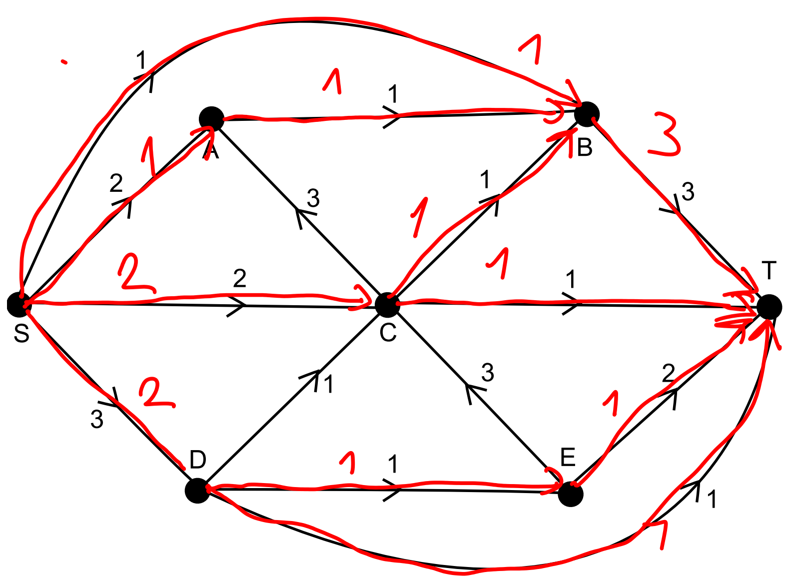


Figure 3: The oriented graph in the questions Q2 and Q3.

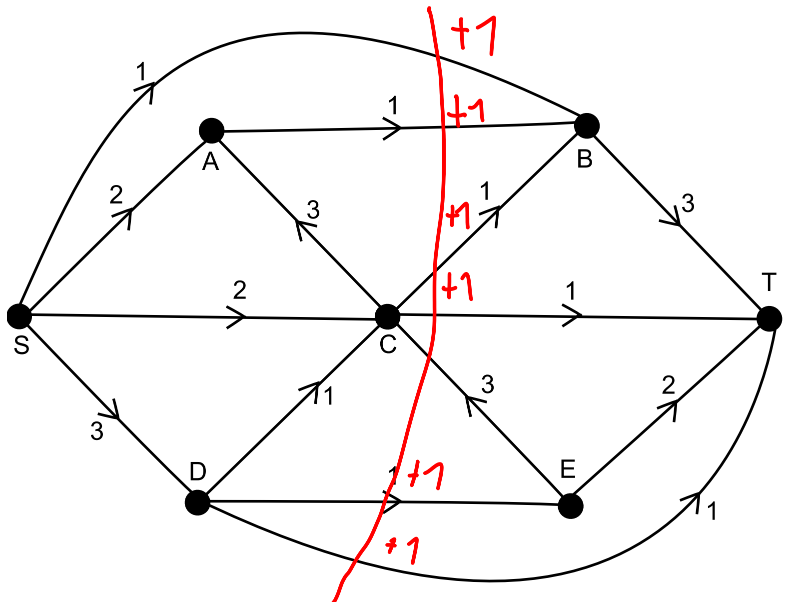
a) Find a maximum flow from the vertex s to the vertex t. *(8 points)*

Solution: See the following flow from s to t of value 6.



b) Find a minimum s,t-cut. *(8 points)*

Solution: See the following s,t-cut of capacity 6.

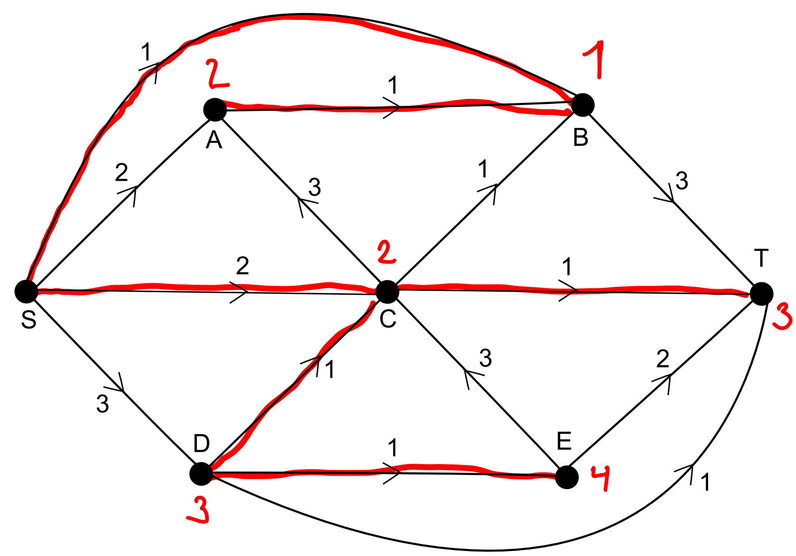


The construction of a flow from s to t of value 6 in (a) and the construction of an s,t-cut with capacity 6 in (b) yields that the flow is a maximum flow and the s,t-cut has minimum capacity. Note that the constructed max-flow / min-cut could be obtained by iteratively applying Ford-Fulkerson Theorem (see Theorem 10.5 in the lecture notes).

Q3 Let $G = (V, E)$ be the simple graph with weights $w : E \to \mathbb{Z}_+$ obtained from the oriented graph depicted in Figure 3 by replacing each oriented edge by a non-oriented one that has the same weight.

a) Find a shortest path spanning tree in G for the vertex S. *(6 points)*

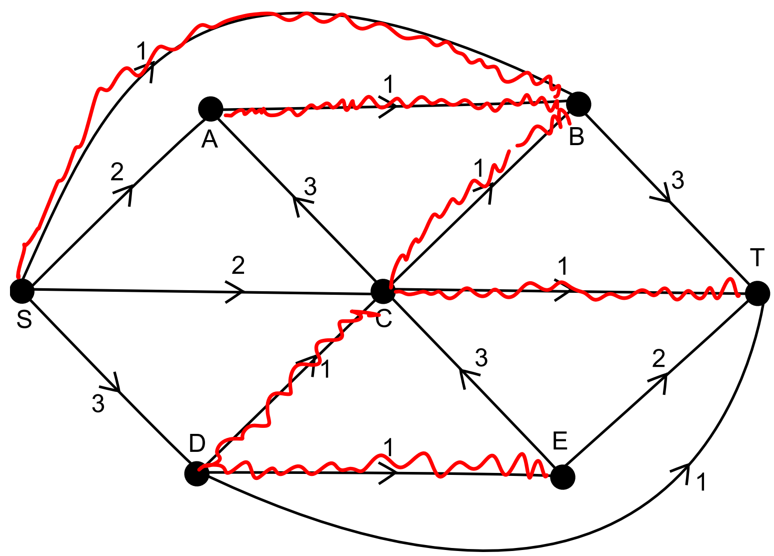
Solution: See the following shortest path spanning tree for S:



Note that this tree can be obtained using Dijkstra's algorithm (see Chapter 5 in the lecure notes).

b) Find a minimum-cost spanning tree in G. *(6 points)*

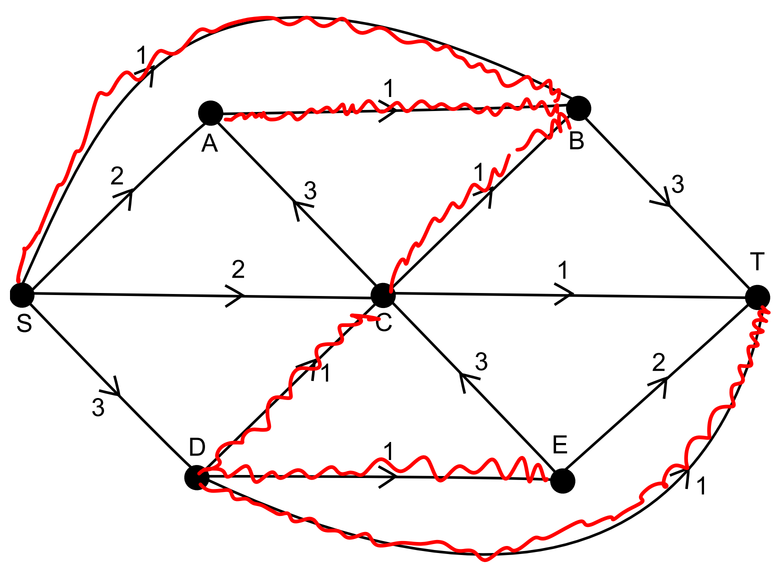
Solution: See a minimum spanning tree of the total weight 6:



Note that this tree can be obtained using Kruskal's algorithm (see Chapter 4 in the lecture notes).

c) Does G have a unique minimum-cost spanning tree? *(6 points)*

Solution: No! See another spanning tree of the total weight 6:



Q4 Let $k \geq 2$ be an integer, and let G be a connected k-regular bipartite graph. Prove that G is 2-connected. *(16 points)*

Solution: Let (A, B) be a bipartition of G. Suppose for contradiction that G contains a cut-vertex v. Without loss of generality, $v \in A$. Let (A_1, B_1) be a bipartition of one of the components of $G - v$ with $A_1 \subseteq A$ and $B_1 \subseteq B$, and let $B_0 := B \setminus B_1$. Let ℓ_0 and ℓ_1 be the number of neighbors of v in B_0 and B_1, respectively. Since G is connected, then $\ell_1 \geq 1$ and hence

$$1 \leq \ell_0 = k - \ell_1 \leq k - 1.$$

Therefore, the number of edges z from B_1 to A_1 in $G - v$ is $|B_1|k - \ell_0$. In particular, $z \not\equiv 0 \mod k$. On the other hand, every vertex in A_1 has degree k, so $z \equiv 0 \mod k$; a contradiction.

Q5 Let G be a simple planar graph. Prove that if G contains no cycle of length five or less, then $\chi(G) \leq 3$. *(16 points)*

Solution: We prove the statement by induction on $|V(G)|$. If $|V(G)| \leq 3$, then the statement is indeed trivially true. In the rest, we assume $|V(G)| \geq 4$.

If G is disconnected, we can use the induction hypothesis on each of its components to find a 3-coloring of each component. The union of these 3-colorings yields a 3-coloring of the whole graph G.

Now suppose G is connected and has a cut-vertex x. Let C be one of the connected components of $G-x$ and let $V_1 := V(C) \cup \{x\}$ and $V_2 := V(G) \setminus V(C)$. It follows from the induction hypothesis that the two subgraphs G_1 and G_2 induced by V_1 and V_2, respectively, are both 3-colorable. Moreover, we can find a 3-coloring c_1 of G_1 and a 3-coloring c_2 of G_2 such that the vertex x is colored both in c_1 and c_2 with the color 1. But then the union of these two 3-colorings again yields a 3-coloring of the whole graph G.

For the rest of the proof, suppose that G is 2-connected and consider any drawing of G. Since G is 2-connected, the boundary of any region R in the drawing consists of a cycle in G. But G has no cycles of length at most 5, so $\ell(R) \geq 6$. Therefore,

$$2|E(G)| = \sum_{R \text{ region}} \ell(R) \geq 6 \cdot Reg(G).$$

This yields that $Reg(G) \leq |E(G)|/3$ and from the Euler's formula we conclude

$$2 = |V(G)| + Reg(G) - |E(G)| \leq |V(G)| - 2|E(G)|/3.$$

A basic algebra manipulation with the last inequality yields that

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| \leq 3|V(G)| - 6,$$

which in-turn implies that G must contain a vertex x of degree 2. By the induction hypothesis applied on $G - x$, we can find a 3-coloring of $G - x$, which is then easily extended to G by coloring x with any color that is not present on its two neighbors.

Q6 Let K_4^- be the 4-vertex graph obtained from K_4 by removing one edge. How many non-isomorphic simple 2-connected graphs $G = (V, E)$ are there with $|V| = 1000$ such that G has no K_4^--minor? *(18 points)*

Solution: The answer is 1 and the only graph G is C_{1000}. If G is 2-connected, then it must contain a cycle. Let C be the longest cycle in G. If G is not isomorphic to C_{1000}, then either C is a Hamilton cycle and G contains at least one chord of C, or, the length of C is at most 999. If C is a Hamilton cycle and G contains at least one chord of C, then clearly C and this chord is a minor of K_4^-.

On the other hand, if C does not cover all the vertices, then let $x \in V(G) \setminus V(C)$. By the 2-connectivity assumption on G, it contains two paths P_1 and P_2 so that $V(P_1) \cap V(P_2) = \{x\}$, $V(C) \cap V(P_1) = \{w_1\}$, $V(C) \cap V(P_2) = \{w_2\}$, the endpoints of P_1 are x and w_1, and the endpoints of P_2 are x and w_2. But then $C \cup P_1 \cup P_2$ is a minor of K_4^-.