Instructions: The exam is 3 hours long and contains 6 questions. The total number of points is 100. Write your answers clearly in the notebook provided. You may quote any result/theorem seen in the lectures without proving it. **Justify all your answers!**

Q1 Let G be the graph depicted in Figure 1.

a) Is G planar? *(4 points)*

b) Does G contain a Hamilton cycle? *(4 points)*

c) Find $\chi(G)$. *(4 points)*

d) Find $\chi'(G)$. *(4 points)*

Q2 Let $\overrightarrow{G} = (V, E)$ be the oriented graph with the two specific vertices s and t and with the capicities $c : E \to \mathbb{Z}_+$ depicted in Figure 2.

a) Find a maximum flow from the vertex s to the vertex t. *(8 points)*

b) Find a minimum s,t-cut. *(8 points)*

Q3 Let $G = (V, E)$ be the simple graph with weights $w : E \to \mathbb{Z}_+$ obtained from the oriented graph depicted in Figure 2 by replacing each oriented edge by a non-oriented edge with the same weight.

a) Find a shortest path spanning tree in G for the vertex s. *(6 points)*

b) Find a minimum-cost spanning tree in G. *(6 points)*

c) Does G have a unique minimum-cost spanning tree? *(6 points)*

Q4 Let $G = (V, E)$ be a graph, and let $u, v, w \in V$ be distinct.

a) Suppose G contains two internally vertex-disjoint paths from u to v and two internally vertex-disjoint paths from v to w. Prove or disprove that G has to contain two internally vertex-disjoint paths from u to w? *(8 points)*

b) Suppose G contains two edge-disjoint paths from u to v, and two edge-disjoint paths from v to w. Prove or disprove that G has to contain two edge-disjoint paths from u to w? *(8 points)*

Q5 Let $G = (V, E)$ be an n-vertex bipartite graph such that $3 \leq \deg_G(v) \leq k$ for every $v \in V$. Prove that G contains a matching of size at least $\frac{3n}{k+3}$. *(17 points)*

Q6 Let G be an n-vertex graph with $\alpha(G) = 2$. Prove that G contains $K_{\lceil \frac{n}{3} \rceil}$ as a minor. *(17 points)*

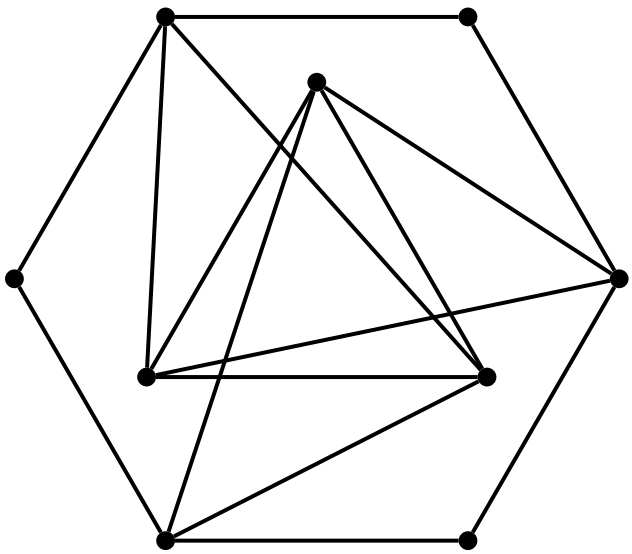

Figure 1: The graph in the question Q1.

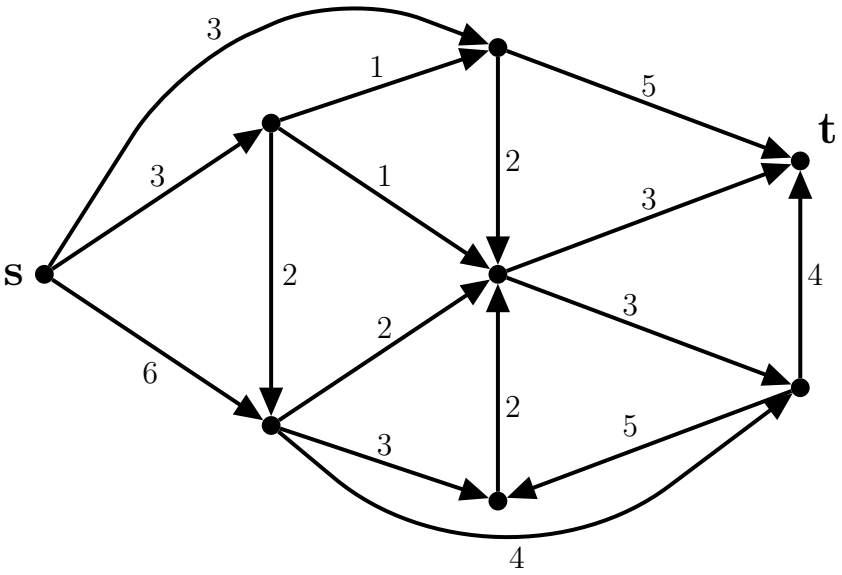

Figure 2: The oriented graph in the questions Q2 and Q3.