Lecture Notes for MATH 350 - Fall 2017

Ralph Sarkis

December 5, 2017

These are my lecture notes taken during the Introduction to graph theory class in fall 2017.

Contents

Course Introduction 1
Introduction to graphs 1
Trees 9
Spanning trees 14
Euler tours 17
Bipartite graphs 20
Matching in graphs 21
Ramsey theory 29
Connectivity of graphs 34
Networks 38
Proper Vertex Coloring 41
Structural graph theory 45
Colorings of planar graphs 50

Course Introduction

This class will be taught by Jan Volec. There will be 10 assignments with one warm-up question and a challenge question on each which will not be graded. Both the midterm and the final will be open-book exams. The weights of the assignments will be 20%, the weights of the midterm will be 20% and the weight of the final will be 60%. The office hours of Jan Volec are on Tuesday from 10:30 to 12:30 in Burnside 1242. There is more information on Jan Volec's website at `http://honza.ucw.cz`.

Introduction to graphs

Definition 1 (Graph)**.** A **graph** G is an ordered pair with a set of vertices V and a set of edges $E \subseteq \binom{V}{2}$[1]. V is normally finite and E is a

[1] Here the binomial notation $\binom{S}{k}$ where S is a set and $k \in \mathbb{N}$ denotes the set of unordered $k-tuples$ with elements in S

set of unordered pair of vertices. E can sometimes be considered as a multi-set so that you can have multiple edges which are the same.

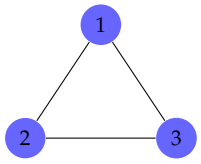

Figure 1: This represents the graph $G = (V, E)$ where $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$

Example 2. See figure 1.

Before actually diving into the formal theory of graphs, we look at simple problems that were solved using graph theory (simple in their formulation, not necessarily their solution).

1. How can you connect houses on an electrical grid with the least amount of wires ?

   The first algorithm to solve this problem was designed by Otakar Borůvka in 1926.

2. What is the shortest distance you can travel while visiting all 647 U.S. colleges exactly once ?

   The tour was found in 2015 by William Cook from UW.

3. Can you cross the seven bridges of Königsberg exactly once ? (See Figure 2.)

   The answer is no and it was proved by Euler. Draw each part of land separated by the river as a node of a graph, and draw the bridges as edges between the nodes. Observe that more than 2 nodes have an odd amount of edges. However, all nodes which are not the start or end of the tour must have an even amount of edges, because if you come on the land from a bridge, you must leave from another bridge. Hence, we can have at most 2 nodes with uneven edges connected, which is not the case in Königsberg puzzle.

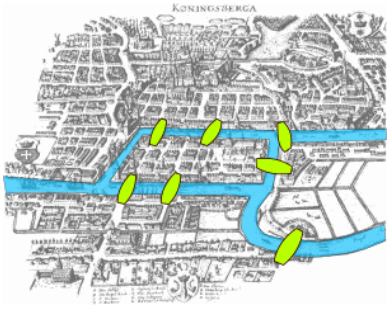
Figure 2: The seven bridges of Königsberg puzzle

4. How can you disconnect 2 railway networks by destroying the least amount of connections between towns ?

   There are secret reports from 1955 that show the solution to this problem on the railway network connecting the Soviet Union to East Europe.

5. Let there be a town with several factories and stores with roads connecting them. There are 3 rules for the roads.

   - No road can connect 2 stores.
   - No road can connect 2 factory.
   - There can be intersection between roads.

   Can there be simultaneously more than 2 factories and more than 2 stores ?

   No, it is not possible.

6. You have eight batteries in your bag and you know that four of them work (you do not know which work or which do not work). You need two charged batteries to power a GPS. What is the best strategy to switch batteries in the device the least amount of time before being sure to have two charged batteries ?

   Make a two groups of three batteries and try every possible combination in each groups (maximum six trials). If no combination worked, the two batteries outside of the group are charged and so you have a total of seven trials. The general solution for this problem is known if you need two batteries but not if you need three.

7. Is every political map 4-colorable ? You color a map with four colors and each country is colored so that no adjacent countries have the same colors.

   This is the first accepted proof that was computer assisted. It is really big and considers lots of cases (that is why a computer was needed). Note that the proof for 6-colorable is almost trivial.

8. In a graph with 20 vertices, you can always find 4 vertices which are either not connected or all connected.

   This is part of Ramsey theory which gives motivation for the Ramsey numbers. The following remark is a result easier to prove than the one above.

   Remark 3. In a complete graph with 6 vertices and 2 colors of edges (red and blue), you can always find a red triangle or a blue triangle.

   Proof. Take an arbitrary node in the graph, call it A. Assume A has at least three blue edges, then between nodes connected connected to A via blue edges there are two possibilities. If there is a blue connection between these nodes, we obtain a blue triangle with A, if there is no blue connection, we obtain a red triangle. If A has less than three blue edges, then it has at least three red edges and the same argument as above holds. □

We will now give several vocabulary definitions that we will often use in the course.

Definitions 4 (Vocabulary of graphs)**.** Let $G = (V, E)$ be a graph.

- Let $e_1, e_2 \in E$ be two edges, we say that they are **parallel** if they join the same pair of vertices.
- Let $e \in E$ be an edge, we say that it is a **loop** if it connects a vertex to itself.

- Let $e = \{u, v\} \in E$ where $u, v \in V$, we say that u and v are the **endpoints** or **ends** of e.
- G is called **simple** if it contains no parallel edges and no loops. It is called loopless if it has no loops.
- **Non-simple** graphs are also called **multi-graph**.
- We use the following notation when working with multiple graphs : $V(G) = V$ and $E(G) = E$.
- Let $u, v \in V$, we say that u and v are **adjacent** if there exists an edge $e \in E$ of which they are the endpoints.
- Let $v \in V$ and $e \in E$, we say that e is **incident** to v or that v is **incident** to e if v is an endpoint of e.
- Let $e_1, e_2 \in E$ be two edges, we say that they are **coincident** if the share a common endpoint.
- The **degree** of a vertex $v \in V$ is defined by the number of edges incident to v. It is denoted $\deg(v)$.

We will take a small break from these definitions to prove a simple lemma and its corollary.

Lemma 5 (Handshaking lemma)**.** *For every graph $G = (V, E)$, the sum of the degrees of all the vertices is even.*

Corollary 6. *The number of vertices with odd degrees is even.*

Proof of corollary. Let $V_0 = \{v \in V \mid \deg(v) \equiv 0 \pmod 2\}$ and $V_1 = \{v \in V \mid \deg(v) \equiv 1 \pmod 2\}$, we decompose the summation of all degrees like so:

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_0} \deg(v) + \sum_{v \in V_1} \deg(v)$$

The summation in V_0 is even and since the left hand side is even, the summation in V_1 must be even. Hence, $|V_1|$ must be even. □

The proof technique we are using here, namely counting the same thing in 2 different ways, is often used in proofs about graphs

Proof of lemma. By the definition of degree, we can infer that each time a vertex v is an endpoint of an edge, its degree is incremented (loop edges count twice for the same vertex). Thus, the summation is equal to the number of endpoints or twice the number of edges (an even number). □

Back to definitions now. When working with simple graphs, we can define really useful matrices.

Definition 7 (Adjacency matrix)**.** Let $G = (V, E)$ be a graph with $V = \{v_1, \cdots, v_n\}$, the **adjacency matrix** is defined by $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Example 8. For the graph in Figure 3., the adjacency matrix is the following:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

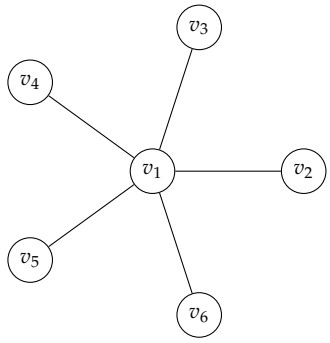

Figure 3: Example of a simple graph

Note that the adjacency matrix is always a $n \times n$ symmetric because when v_i is adjacent to v_j, the other way is true as well

Definition 9 (Incidence matrix)**.** Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$, the **incidence matrix** is defined by $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 & e_j \text{ is incident to } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example 10. For the graph in Figure 3., we denote e_1 to be the edge going from v_1 to v_2 and increment the subscript counter clockwise, the incidence matrix is the following:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The intuition for the adjacency matrix is that is shows which edges are connected to together[2]. Since, we are familiar with matrices from linear algebra, there are natural questions we can ask.

[2] I did not find a nice intuition for the incidence matrix, maybe we will see more of it in the class

Question 11. *What does $A \cdot A$ mean ? (where A is the adjacency matrix and $\cdot$ is the usual matrix product)*

Answer 12. Obviously, it is a $n \times n$ matrix and we have $A^2 = (a_{ij})$, where a_{ij} is the number of walks from v_i to v_j with distance 2. A consequence of that is that the diagonal is the degrees of the vertices, namely, $a_{ii} = \deg(v_i)$.

Question 13. *What does $B \cdot B^T$ mean ? (where B is the incidence matrix)*

Answer 14. We end up with a $n \times n$ matrix and we have

$$B \cdot B^T = A + \operatorname{diag}(\deg(v_1), \ldots, \deg(v_n))$$

We talked about a walk in the first answer and you can actually have an intuition about what a walk is on a graph but we will define it in a formal way along with other similar definitions.

Definition 15 (Walk)**.** Let $G = (V, E)$ be a graph, a **walk** on G is a sequence $\{v_0, e_1, v_1, \ldots, e_k, v_k\}$ where $\forall i \in \mathbb{N}, v_i \in V, e_i \in E$ with the property that $\forall i \geq 1, e_i = \{v_{i-1}, v_i\}$.

Definition 16. $\sim_G^W$[3] is a relation on $V(G)$ with $u \sim_G^W v$ if and only if there exists a walk on G from u to v.

[3] We use the W superscript to specify that is a walking relation

Definition 17 (Trail)**.** Let $G = (V, E)$ be a graph, a **trail** on G is a walk on G with the property that no edge is repeated.

Definition 18. $\sim_G^T$ is a relation on $V(G)$ with $u \sim_G^W v$ if and only if there exists a trail from u to v.

Definition 19 (Path)**.** Let $G = (V, E)$ be a graph, a **path** on G is a walk on G with the property that no vertex is repeated[4].

[4] A consequence is that no edge is repeated so every path is a trail

Definition 20. $\sim_G^T$ is a relation on $V(G)$ with $u \sim_G^W v$ if and only if there exists a trail from u to v.

The next proposition will show that these three relations are redundant and that we will only need one of them.

Proposition 21. *All these relations are the same, namely, the existence of walk guarantees the existence of a trail which guarantees the existence of a path which guarantees the existence of a walk.*

Before proving it, we prove a corollary which motivates the definition of these relations.

Corollary 22. *For any graph G, the walk, trail and path relations are equivalence relations on $V(G)$.*

Proof. Let $G = (V, E)$ be a graph and $u, v, w \in V$ be vertices. Using the proposition, we only need to prove it for a walk. The walk relation is symmetric because if there exists a walk from u to v, the reversed sequence is a walk from v to u. It is also reflexive because u is always a valid walk from u to u. To prove transitivity, assume that $u \sim_G^W v$ and $v \sim_G^W w$, then there exists sequences $\{u, e_1, v_1, \ldots, e_m, v\}$ and $\{v, e_1', v_1', \ldots, e_n', w\}$ that represent walks on G. Indeed, we can create a new walk $\{u, e_1, v_1, \ldots, e_m, v, e_1', v_1', \ldots, e_n', w\}$ from u to w. □

Proof of the proposition. We will show that if a walk exists between $u, v \in V(G)$ for an arbitrary graph G, a path also exists from u to v. The other directions we need for equivalence between walk, trail and path are all trivial. Suppose that there exists a walk from u to v, then we can find the smallest walk $W = \{w_0, e_1, w_1, e_2, \ldots, e_n, w_n\}$ where

$w_0 = u$ and $w_n = v$[5]. We claim that this walk is a path. Assume that for some $1 \leq i < j \leq n$, $w_i = w_j$. Then, we can construct a smaller walk $W' = \{w_0, e_1, w_1, \ldots, e_i, w_i, e_{j+1}, w_{j+1}, \ldots, w_n\}$ that still goes from u to v, contradicting our choice of W. Since, W does not have a repeated vertex, it is a path. □

[5] A formal argument for that would be to use the well-ordering principle on the set of sizes of walks from u to v

Definition 23 (Connected components)**.** The equivalence classes defined by the $\sim_G$ relation are called **connected components**. See Figure 4. for an example. For some graph G, the number of connected components in the graph is denoted $\text{comp}(G)$. We say that a graph G is **connected** if $\text{comp}(G) = 1$.

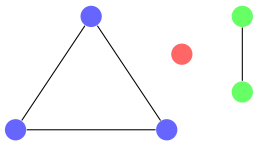

Figure 4: The connected components of this graph each have different colors

To formalize some of the concepts we described above, we sometimes use the notion of subgraphs.

Definition 24 (Subgraph)**.** Let $G = (V, E)$ be a graph. We say that a graph $H = (V', E')$ is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$. In other words, all the vertices in H must be in G and all the edges in H must be in G and incident to the vertices in H. We say that a subgraph H of G is induced if $E(H) = E(G) \cap \binom{V(H)}{2}$.

Remark 25. For a graph $G = (V, E)$, a vertex-maximal subgraph of G that is connected is a connected subgraph for which you cannot add a node from G and still get a connected subgraph. Notice that any vertex-maximal subgraph of G is a connected component of G.

Here we present some common examples of graphs.

Examples 26.

1. The null graph is a graph with no vertices and no edges : $G = (\emptyset, \emptyset)$.
2. The empty graph on n vertices is $E_n = (V, \emptyset)$ where $V = \{1, \ldots, n\}$.
3. The complete graph on n vertices is $\overline{E_n} = K_n = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \binom{V}{2}$.
4. A path on n vertices is $P_n = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{e_1, \ldots, e_{n-1}\}$ with $e_i = \{i, i+1\}$ for $i \in \{1, \cdot, n-1\}$.
5. A cycle on n vertices is $C_n = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{e_1, \ldots, e_n\}$ with $e_i = \{i, i+1\}$ for $i \in \{1, \cdot, n-1\}$ and $e_n = \{n, 1\}$.

Remark 27. When we talk about the length of paths or cycles, we will refer to the number of edges in the path/cycle.

When working with graphs, we sometimes need to compare them and say that they are "equal". We formalize this with the notion of isomorphism.

Definition 28 (Graph isomorphism)**.** Let G and H be graphs, we say that G is **isomorphic** to H[6] and denote it $G \simeq H$ if there exists a bijection $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$. In simpler terms, we just want to be able to rearrange the representation of G so that it looks like the representation of H.

[6] We often make an abuse of vocabulary and just say that G is H

Remark 29. In this course, when we talk about any graph, we are in fact talking about all the graphs that are isomorphic to it (i.e: we never make distinctions between graphs that are isomorphic).

We quickly go back to the notion of connected components and prove an interesting result.

Definition 30. Let $G = (V, E)$ be a graph. For some $e \in E$, we denote $G - e$ to be the graph G with the edge e removed. Namely, $G - e = (V, E')$ with $E' = E \setminus \{e\}$.

Definition 31 (Cut edge)**.** An edge $e \in E(G)$ is a **cut edge** if

$$\text{comp}\,(G - e) = \text{comp}\,(G) + 1$$

.

Lemma 32. *Let G be a graph, $e \in E(G)$ is a cut edge if and only if there is no cycle in G containing e.*

Proof. ($\Rightarrow$) Denote $e = \{u, v\}$, we know that $G - e$ has one more connected component than G. This implies that in $G - e$, the vertex u is in a different component than v. Assume towards a contradiction that there is a cycle containing e, we can write it as $C = \{v_1, e_1, v_2, \ldots, v_k, e_k, v_1\}$ where $v_1 = u$, $v_k = v$ and $e_k = e$. Now, we can construct a path $P = \{v_1, e_1, \cdots, v_k\}$ which is valid in $G - e$ but this contradicts the fact that u and v are not connected.

($\Leftarrow$) If e is not a cut edge, then u and v are still connected in $G - e$. Thus, we can find a path from u to v in $G - e$. If you add e to the path in G, you form a cycle that contains e. By contrapositive, this shows that if there is no cycle containing e, e is a cut edge. □

We can do similar work with a vertex instead of an edge.

Definition 33. Let $G = (V, E)$ be a graph. For some $v \in V(G)$, we denote $G - v$ to be the graph G with the vertex v removed as well as all the edges incident to v. You can also view it as the induced subgraph with v removed.

Definition 34 (Cut vertex)**.** A vertex $v \in V(G)$ is a **cut vertex** if

$$\text{comp}\,(G - v) > \text{comp}\,(G)$$

Remark 35. There is no analog to the lemma above for a cut vertex.

Trees

In this section, we will present a class of graphs we call trees and prove some result about them.

In order to define a tree, we first look at some graphs which we want to call trees and some graph we do no want to call trees and try to find formal definitions for them. Then, we will prove that all these definitions are equivalent. Here is a list of definition that seem to fit our purpose. A tree is a graph G such that :

I am really lazy but there should be some diagram of trees and some diagram of non-trees (look online if you really want those examples)

1. G is connected and contains no cycle
2. $\forall e \in E(G)$, e is a cut-edge
3. G is connected and every trail in G is a path
4. Between any two vertices there is a unique path.
5. Maximal graph with respect to adding edges that has no cycle
6. G is connected and $|V(G)| = |E(G)| + 1$

The first of these definitions is the one that is most often used and it also is one of the clearest. Thus, we will use it to define trees and prove that all the others are equivalent.

Definition 36 (Tree)**.** A **tree** is a connected graph with no cycle.

Proposition 37. *All the definitions given above are equivalent.*

Proof. (1 $\Leftrightarrow$ 2) follows trivially from Lemma 32.

(1 $\Leftarrow$ 3) Let $T = \{v_0, e_1, v_1, \ldots, e_k, v_k\}$ be a trail with $v_i = v_j$ for some $i < j$, without loss of generality, suppose that this is a pair such that $j - i$ is minimal. Then, $\{v_i, e_{i+1}, \ldots, e_j, v_j\}$ is a cycle in G and we have a contradiction.

(1 $\Rightarrow$ 3) Suppose there is a cycle in G, then, there is a trail which is not a path and we again have a contradiction. □

Before doing the rest, we will give the definition for a leaf and 2 results that we will use to make our proofs smaller.

Definition 38 (Leaf)**.** Let G be a graph, $v \in V(G)$ is called a **leaf** if $\deg(v) = 1$.

Lemma 39. *Every tree with at least two vertices has at least two leafs.*

Proof. Since there is two vertices, the smallest path has length 1 and contains two ends (this is important for te last part). Take the longest path in the graph, the ends of this path must have degree one. Indeed, if one end was connected to another vertex, it would either be a vertex

not in the path, meaning there exists a longer path, or a vertex in the path, meaning there is a cycle in the tree. Both lead to a contradiction. The two ends in this path are the two leafs in the tree. □

Corollary 40. *Let G be a graph. For any leafs $v \in V(G)$, G is tree if and only if $G - v$ is a tree.*

Proof. ($\Rightarrow$) Since v has degree 1 in G, it cannot be part of a cycle, and since all vertices in $G - v$ are connected to the only vertex adjacent to v, v is connected to all of G. We get that G is connected and has no cycle. ($\Leftarrow$) Since v has degree 1, it cannot be part of a path. Thus, $G - v$ is still connected because no path between vertices passed through v. Moreover, removing a vertex cannot create a cycle in $G - v$. We get that $G - v$ is connected and has no cycle. □

Now, we continue the proof of the proposition.

Proof of proposition (continued). We will do an induction on the number of vertices (denoted n) and prove that 1 implies 4, 5 and 6. When $n = 1$, 4,5 and 6 are trivially true for a tree with a single vertex. Let T be a tree on $n > 1$ vertices, it contains a leaf v by the lemma and $T - v$ is a tree by the corollary. By induction hypothesis, 4, 5 and 6 are satisfied in $T - v$.

From every vertex in $T - v$, there is an unique path to v because v has only one adjacent vertex and it has unique paths to every vertex in $T - v$. Moreover, since v cannot be part of path in which it is not one of the ends, 4 is satisfied.

Suppose you can add an edge such that the graph still has no cycle, the edge must be incident to v because 5 is satisfied for $T - v$. However, that would mean the edge would be part of a new path from v to the other endpoint of the edge and this contradicts 4. Thus, 5 is also satisfied.

Lastly, 6 is clearly satisfied because we are adding one edge and one vertex to the graph.

(4 $\implies$ 1) If there is a unique path between any two vertices, in particular, there is a path so it is connected. Moreover, if there was a cycle in G, you could decompose it into two different path between the same vertices.

(5 $\implies$ 1) We already have that it has no cycle. Suppose $\exists u, v \in V(G)$ such that u is not connected to v. If we add the edge $\{u, v\}$, we will not create a cycle because there is no other path from u to v. Hence, there is a contradiction and G is in fact connected.

(6 $\implies$ 1) We already have that it is connected. If G has a vertex of degree 1, then take $G - v$, it has $|V| - 1$ and $|E| - 1$ edges, so 6 is still satisfied. By continuing to remove vertices, we will either end up with $|G'| = 1$ meaning that we found no cycles (since we

were able to keep removing vertex of degree 1) or with $|G'| > 2$ and $\deg(v) \geq 2, \forall v \in V(G')$. Suppose that the latter is true, then $2|E| = \sum_{v \in V(G')} \deg(v) \geq 2|V|$ which implies $|E| \geq |V|$ which contradicts 6. Hence, the first case is the only one possible and there cannot be any cycle. □

Our next goal is Cayley's formula but we will explore similar ideas before actually stating it, getting away from the realm of the trees for a bit. Here is one natural question one might ask about graphs.

Question 41. *How many isomorphism classes of simple graphs on n vertices are there ?*

We will denote this number g_n. While looking for an answer to this question, you might consider a question that seems simpler.

Question 42. *How many labeled simple graphs on $\{1, \ldots, n\}$ are there ?* [7]

[7] By labeled, we mean that isomorphic graphs are not considered to be the same

Denote this number h_n. Since there is $\binom{n}{2}$ possible edges in the graph G and each edge is either in $E(G)$ or not in $E(G)$, we clearly have $h_n = 2^{\binom{n}{2}}$.

Remark 43. Clearly, we have $g_n \geq h_n$. Moreover, observe that each isomorphism class of the labeled graphs have a size of at most $n!$ graphs. Thus, we get $h_n \geq \frac{g_n}{n!}$. If we consider the upper bound $n! < n^n$ and the formula $2^{\binom{n}{2}} = 2^{n\left(\frac{n-1}{2}\right)}$, we obtain the following :

$$\begin{aligned} 2^{\frac{n^2}{2}} \approx 2^{n\left(\frac{n-1}{2}\right)} \geq h_n &\geq \frac{2^{\binom{n}{2}}}{n^n} \\ &\geq 2^{\binom{n}{2} - n\log(n)} \\ &= 2^{n\left(\frac{n-1}{2} - \log(n)\right)} \\ &\approx 2^{\frac{n^2}{2}} \end{aligned}$$

We can see that asymptotically g_n and h_n are equal.

We will consider that a satisfying answer for our initial question but you might already see that there are some improvements that can be made. Instead of detailing them, we will ask a new question that will get us closer to Cayley's formula.

Question 44. *How many isomorphism class on trees on n vertices are there ?*

Denote this number u_n. Once again, we ask a simpler question before giving an answer for this one.

Question 45. *How many labeled trees on $\{1, \ldots, n\}$ are there ?*

Denote this number t_n. This question is answered by Cayley's formula which states that $t_n = n^{n-2}$. Before giving a proof, we relate t_n to u_n in the same way as we related g_n to h_n. Clearly, we have $t_n \geq u_n \geq \frac{t_n}{n!}$. Now, if we try to end up bounding $n!$ with n^n again, we end up with $u_n \geq \frac{1}{n^2}$ which gives that for $n > 2$ there is at least zero isomorphism class of trees on n vertices. Compared to our previous result, this is really underwhelming and we will try to do better. Sterling's approximation is the best one for $n!$ and it is the following :

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

We obtain the following lower bound :

$$u_n \geq \frac{n^{n-2}e^n}{n^n\sqrt{n}\sqrt{2\pi}} = \frac{e^n}{n^{\frac{5}{2}}\sqrt{2\pi}}$$

Remark 46. It is known that u_n behaves like $C\frac{\alpha^n}{n^{\frac{5}{2}}}$ where $\alpha \approx 2.956$ and $C \approx 0.535$.

We still need one little thing before proving Cayley's formula.

Joke 47. There once was a hunter that lived in one of the biggest forests in Quebec. He woke up one day with the idea to kill every deers in his forest. But before he could kill them all, he needed to count them. He was not really good at even the most basic mathematics, so he went to ask his mathematician friend how many deers there were in his forest. Surely, the mathematician quickly came up with an answer : "There are 1729 deers in your forest."

The hunter was really impressed and asked his friend how he counted. To that, the mathematician answered : "Well it is pretty easy, I counted their legs and divided by four."

For Cayley's formula, we will use the same technique as the mathematician in this joke to count the trees. Namely, we will count something else which is simpler to count and find the relation between the number of trees and this thing. In fact, we will go even deeper with this method and that is why we will define some new objects before the proof.

Theorem 48 (Cayley's formula)**.**

$$t_n = n^{n-2}$$

Definition 49 (Rooted tree)**.** A **rooted tree** T is a tree plus one special vertex $v \in V(T)$ which is called the root. There are no restrictions to where this vertex is in the tree.

Definition 50 (Orientation)**.** An **orientation** of a graph G is a function $o : E(G) \to V(G)$ such that $\forall e \in E(G), o(e) = v$ where v is one of

the endpoints of e. You can see it as adding one arrow to each edge to show which end it is pointing at.

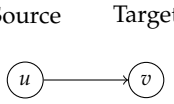

Figure 5: Example of an oriented edge. By convention, $o(e)$ is the target.

Definition 51 (Out-rooted orientation)**.** An orientation of a rooted tree is called **out-rooted** if every vertex except the root is the target of exactly one edge.

Lemma 52. *For every rooted trees, there exists a unique out-rooted orientation.*

Proof. Orient the edges from the root towards the neighbors of the root. Then, orient the edges from vertices at distance i from the root towards the vertices at distance j from the root. This is a out-rooted orientation although the proof is left to the reader. Now, we will show that this is the unique orientation. Suppose o_1 and o_2 are two out-rooted orientation of some rooted tree T with a root r. If $|V(T)| = 1$ there are no edges so o_1 is vacuously equal to o_2. Suppose that $|V(T)| > 1$, then take a leaf $v \in V(T) \setminus \{r\}$ (we know there exists one from Lemma 39). Denote w to be the only neighbor of v and $e = \{u, w\}$ to be the edge that connects them. Clearly, we have $o_1(e) = o_2(e) = v$ because if w was the target, v would not have any incoming edge. Define $o_1' = o_1|_{E(T)\setminus\{e\}}$ and $o_2' = o_2|_{E(T)\setminus\{e\}}$. Clearly, they are out-rooted orientation of $T - v$ and by induction hypothesis, we must have $o_1' = o_2'$. However, this must mean that $o_1 = o_2$ because we already saw that they agree on e which was the only edge we removed. □

Proof of Cayley's formula. Denote t_n' to be the number of rooted trees on $\{1, \ldots, n\}$. There is n possible choices for the root so we have $t_n' = n \cdot t_n$. Thus, we will show that $t_n' = n^{n-1}$. We already know that the number of trees on $\{1, \ldots, n\}$ with an out-rooted orientation is the same as t_n', so our formula does not change here. Denote t_n'' to be the number of trees on $\{1, \ldots, n\}$ with an out-rooted orientation and an ordering of the edges. Clearly, there are $(n-1)!$ possible choices for this ordering ($|S_{n-1}|$), so we obtain $t_n'' = t_n'(n-1)!$. Our goal is now to show that $t_n'' = n^{n-1}(n-1)!$.

We will describe a procedure to build the trees counted by t_n'' and then we will count the number of ways to run the procedure. The procedure will maintain two key properties :

- Whatever is built at each step is acyclic.
- Every vertex will be the target of at most one edge.

The zeroth step is to start with an empty graph on $\{1, \ldots, n\}$. The next $n-1$ steps are the following:

1. Choose a vertex $s \in \{1, \ldots, n\}$

2. Choose a vertex $t \in \{1, \dots, n\}$ such that $t \neq s$, t has no incoming edge and adding the edge $\{s, t\}$ will not create a cycle.

3. Add the edge $\{s, t\}$ with orientation $o(\{s, t\}) = t$.

Notice that if the procedure is run with at least one different choice for s or t you get a different tree. Hence, we need to count the number of choices we have. Clearly, we always have n choices for s. Observe that a the $k+1$th step, the graph has $n-k$ connected components. Using the invariants we can see that each of these components have exactly one vertex with no incoming edge. Since connecting two components with an edge cannot create a cycle, we have $n-k$ choices for t. Hence, the total number of choices is $t''_n = \prod_{k=1}^{n-1} n(n-k) = n^{n-1}(n-1)!$. □

Spanning trees

While trees can be studied on their own, it is sometimes useful to study trees embedded in a graph.

Definition 53 (Forest)**.** A **forest** is a graph with no cycle (also called acyclic). In other words, each connected components of G are trees.

Definition 54. Let G be a graph, a **forest in** G is an acyclic subgraph of G.

Definition 55 (Spanning subgraph)**.** Let $G = (V, E)$ be a graph, and H be a subgraph of G. We say that H is a **spanning subgraph** of G if $V(H) = V$.

Definition 56 (Spanning tree)**.** Let G be a graph, a **spanning tree** of G is a spanning subgraph of G that is a tree.

Proposition 57. *If G is connected, then G has a spanning tree.*

Proof. Let G be a graph, take a connected spanning subgraph of G which is minimal with respect to the number of edges. We know at least one connected spanning subgraph exists because G itself satisfies these properties. Call this subgraph T and we claim that it is a tree. Suppose that there exists a cycle $\{v_0, e_1, v_1, \dots, e_k, v_0\}$ in T. Then, $T - e_k$ is still a connected spanning subgraph of G and has one less edge than T. We get a contradiction. □

Definition 58 (Fundamental cycle)**.** Let G be a graph and T be a spanning tree of G. Moreover, take one edge $e \in E(G) \setminus E(T)$. We define the **fundamental cycle** of G with respect to T and e to be the unique cycle in $T + e$.

Proposition 59. *In the same setting as last definition, take any edge f in the fundamental cycle with respect to T and e. Then, $T_p = (T + e) - f$ is a spanning tree.*

Proof. There is only one cycle in $T + e$, by removing f, we clearly do not create any cycle nor disconnect the graph, hence T_p is still acyclic and connected. Furthermore, we did not remove edges so $V(T_p) = V(T) = V(G)$ and we see that T_p is indeed spanning. □

One meaningful problem related to spanning trees is the minimum spanning tree problem which is well studied in computer science and especially in algorithm design. We have one definition to make before stating the problem.

Definition 60 (Weighted graph)**.** A **weighted graph** is a triple (V, E, w) where (V, E) is a graph and $w : E \to \mathbb{R}$ assigns weights to each edge. For any subgraph H of a weighted graph G, $w(H) = \sum_{e \in E(H)} w(e)$ is the weight of H.

The minimum spanning tree problem or MST for short asks what is the spanning tree with smallest weight for some connected weighted graph G. We will give an algorithm to find the MST of such a graph. The input of this algorithm is a weighted graph and the output is a spanning tree of this graph with minimal weight.

Denote the input $G = (V, E, w)$. The first step of the algorithm is to initialize $T = (V, \emptyset)$. Then, we sort the edges in E according to their weight to obtain an ordering $\{e_1, \ldots, e_m\}$ with $w(e_i) < w(e_j)$ for all $i < j$. Next, we iterate from $i = 1$ to $i = m$. At each iteration, we add e_i to the edges of of T if $T + e_i$ is still a tree. At the end of these iteration, we will obtain a MST of G. It is clear that T is a spanning tree but it is maybe not so clear that it is minimal. What follows is the proof of that.

Correctness of Kruskal's algorithm. We will prove it by induction on the number of edges of T. At the start when T has no edges, there clearly exists a MST of G which contains T. We will show that this property holds whenever T gets a new edge implying that the final T is a MST of G. Suppose that at some step in the algorithm, we are about to add some edge e to T and that M is a minimal spanning tree containing T. If e is in M, then clearly M contains $T + e$. If e is not in M, then $M + e$ has a cycle. Moreover, there exists an edge f in the cycle and not in T (if there were no such edge, e would have created a cycle in T). We obtain that $M - f + e$ is a spanning tree, this means that its weight is no less that the weight of M because M is a MST. In addition, $w(f) < w(e)$ is not possible or the algorithm would have chosen to add f before e. Hence, we must have $w(f) = w(e)$ and $M - f + e$ is in fact a MST containing T. □

We now explore a problem that is closely related to the MST problem. In a simple graph, how can you find the shortest path between two vertices ? If we consider an unweighted graph, the solution is pretty simple. Run a breadth-first search algorithm on the graph starting at the source vertex and stop when the target vertex is reached for the first time. The path taken to the target vertex must be of shortest length. Next, we answer the same question in the setting of a weighted graph but we give a definition first.

Definition 61 (Distance in a graph)**.** Let $G = (V, E, w)$ be a weighted graph with $w : E \to \mathbb{R}^+$. Define the function of **distance** in G like so :

$$\text{dist}_G : V \times V \to \mathbb{R}^+, (u, v) \mapsto \min_{P \in \{\text{paths from u to v}\}} w(P)$$

Remark 62. We a require that the weights are positive because it is extremely hard to solve the question otherwise and most real life settings only need positive weights. Also, note that in the MST problem, we said that the image of w was in the reals but it is sufficient for it to be in a linearly ordered set.

Proposition 63. *The distance function in a weighted graph is a metric.*

Proof. Not in the scope of this class. □

Once again, we present an algorithm that will find the shortest path between two vertices in a graph. Without loss of generality, we assume that this graph is connected. Let $G = (V, E, w)$ be a connected weighted graph, we want to find the shortest path between s and t, both in vertices in V.

We describe Dijkstra's algorithm. This algorithm will build a tree T_s such that $\forall v \in T_s$, $\text{dist}_G(s, v) = \text{dist}_{T_s}(s, v)$. In particular, the unique path in T_s from s to t, will be a shortest path in G. Each step of the algorithm will add one edge to T_s. Note that since we will stop when t is reached, T_s is not necessarily a spanning tree.

In step 0, the tree is initialized to $T_s^0 = (\{s\}, \emptyset)$. Next, we repeat this step while $t \notin V(T_s^k)$. Choose an edge $f = \{u, v\} \in E$ such that $u \in T_s^{k-1}$, $v \notin T_s^{k-1}$ and $\text{dist}_{T_s^{k-1}}(s, u) + w(f)$ is minimal. Add the edge f to get $T_s^k = T_s^{k-1} + f$.

Correctness of Dijkstra's algorithm. In order to prove that this algorithm achieves the result we want, we will show that the following invariant holds at any step k :

$$\forall v \in T_s^k, \text{dist}_{T_s^k}(s, v) = \text{dist}_G(s, v)$$

When $k = 0$, this is clearly true since the distance from a vertex to itself is 0. Let $k > 0$ and assume that this invariant holds for all

the steps before k. Call v the new vertex added at step k, namely, $v \in T_s^k \setminus T_s^{k-1}$. Denote $f = \{x, v\}$ to be the edge that was added at this step. We claim that for any path $P = \{u_0, e_1, u_1, \ldots, e_l, u_l\}$ with $u_0 = s$ and $u_l = v$, $w(P) \geq \text{dist}_{T_s^k}(s, v)$. Let u_j be the first vertex in P that is not in T_s^{k-1}, we know it exists since $v \notin T_s^{k-1}$. We obtain the following:

$$\begin{aligned}
w(P) &\geq \text{dist}_G(s, u_{j-1}) + \sum_{i=j}^{l} w(e_i) \\
&\geq \text{dist}_G(s, u_{j-1}) + w(e_j) \\
&\geq \text{dist}_{T_s^{k-1}}(s, u_{j-1}) + w(e_j) && \text{By induction hypothesis} \\
&\geq \text{dist}_{T_s^{k-1}}(s, x) + w(f) && \text{By the steps in the algorithm} \\
&= \text{dist}_{T_s^k}(s, v)
\end{aligned}$$

□

Euler tours

In this section, we will talk about multigraphs. The arguments we will make are valid for graphs as well but they require some details which we do not want to spend too much time on. We start with some simple definitions.

Definition 64 (Tour)**.** A **tour** in G is a walk $\{v_0, e_1, v_1, \ldots, e_k, v_k\}$ such that $\forall i \neq j, e_i \neq e_j$.

Definition 65 (Closed walk/tour)**.** A walk/tour is said to be **closed** if $v_0 = v_k$, namely, the start and end vertex is the same.

Definition 66 (Eulerian tour)**.** A tour is said to be **Eulerian** if $E(G) = \{e_1, \ldots, e_k\}$ and $V(G) = \{v_1, \ldots, v_k\}$.

We would like to find some necessary and sufficient condition for a graph to contain an Eulerian tour. You can try to draw some graphs and figure out a condition but you might remember from the Königsberg bridges problem that you should have at most two vertices of odd degree for a graph to contain an Eulerian tour. We state a stronger statement and prove our result as a corollary.

Theorem 67. *A multigraph G contains a closed Eulerian tour if and only if G is connected and there is no vertices of odd degree.*

Corollary 68. *A multigraph G contains an Eulerian tour if and only if it is connected and contains at most two vertices of odd degree.*

Proof of corollary. ($\Rightarrow$) This direction does not use the theorem. If there is an Eulerian tour, then all the vertices of G are in the same

walk and so G must be connected. Moreover, all the vertices which are not the endpoints of the walk must have even degree and we have at most two distinct endpoints so we are done.

($\Leftarrow$) If the number of vertices of odd degree is 0, we can use the theorem. There cannot be only one vertex of odd degree so the case where there is two vertices of odd degree is the only one left. Call u, w the vertices with odd degree. Let $G' = G + \{u, w\}$[8], we can now use the theorem to find a closed Eulerian tour $T = \{v_0, e_1, \ldots, e_i, v_i, \ldots, e_k, v_k\}$ where, without loss of generality, $v_0 = v_k = u$ and $e_i = \{u, w\}$. Define the sub-walks $T_1 = \{v_0, e_1, \ldots, v_{i-1}\}$ and $T_2 = \{v_i, \ldots, e_k, v_k\}$. Clearly, T_1 and T_2 are walks in G, and T_2T_1 is an Eulerian tour in G. □

[8] This is where having a multigraph is useful since we do not need to handle the case where the edge is already there

Proof of theorem. ($\Rightarrow$) As in the corollary, G is connected and all the vertices which are not endpoints of T have even degree. Moreover, since the start and endpoint is the same, it must have even degree as well.

($\Leftarrow$) Let $T = \{v_0, e_1, \ldots, e_k, v_k\}$ be the longest tour in G (with respect to the number of edges), we claim that T is closed and Eulerian.

First, assume towards a contradiction that T is not closed. Look at the graph $H = (V, E(T))$. The vertices v_0 and v_k must have odd degrees in H since $v_0 \neq v_k$. Since we know $\deg(v_0)$ is even in G, there exists an edge $f = \{v_0, x\} \in E(G) \setminus E(T)$. Clearly, xfT is a tour with more edges than T, so we obtain that T must be closed.

Next, assume towards a contradiction that there exists an edge $f = \{u, w\} \in E(G) \setminus E(T)$. We consider two cases. If at least one vertex of f is somewhere in T, without loss of generality, $u = v_i$ for some $0 \leq i \leq k$, define the sub-walks $T_1 = \{v_0, e_1, \cdots, v_{i-1}\}$ and $T_2 = \{v_i, \ldots, v_k\}$. We can see that $T' = v_iT_2v_kT_1v_ifw$ is a longer tour so we get a contradiction. Conversely, if f has no endpoint in T. Let P be the shortest path from some vertex in T to some vertex not in T. Clearly, P is just an edge $\{x, y\}$ with $x \in V(T)$ and $y \notin T$ and this leads to a contradiction like in the first part of this proof. Since all the edges are in T and G is connected, all the vertices must also be in T so T is Eulerian. □

A notion related to Eulerian tours is Hamiltonian paths.

Definition 69 (Hamiltonian paths)**.** A path is said to be **Hamiltonian** if it visits all vertices.

Definition 70 (Hamiltonian cycle)**.** A Hamiltonian path with its starting point being its endpoint is called a **Hamiltonian cycle**.

Unfortunately, as of today, there is no necessary and sufficient condition for a graph to contain a Hamiltonian cycle. However, there is a sufficient condition.

Theorem 71 (Ore's theorem). *Let $G = (V, E)$ be a graph with $n = |V| \geq 3$. Suppose that for every pair $u, w \in V$ such that $\{u, w\} \notin E$, $\deg(u) + \deg(w) \geq n$, then G contains a Hamiltonian cycle.*

Proof. Fix n, the number of vertices, we will prove this by induction on the number of edges missing from K_n, namely, an induction on $z = \binom{n}{2} - |E|$. When $z = 0$, the graph is K_n which clearly contains a Hamiltonian cycle. Let $z > 0$ and assume that any graph with $z - 1$ edges removed that satisfies the property in the theorem contains a Hamiltonian cycle. Then, consider a graph G with z edges removed, still satisfying the property. We know there exists an edge $e = \{u, w\} \notin E(G)$. Consider the graph, $H = G + e$, it is trivial to show that H also satisfies the property and has $z - 1$ edges removed, so by our induction hypothesis, it contains a Hamiltonian cycle C_H. If C_H does not contain the edge e, we are done since C_H is also a Hamiltonian cycle in G. On the other hand, if C_H contains e, then, we can write $C_H = \{v_1, e_1, v_2, \ldots, v_n, e, u\}$ where $v_0 = u$ and $v_n = w$.

We claim that there exists an integer $i \in \{3, \ldots, n-1\}$ such that v_1 is adjacent to v_i and v_n is adjacent to v_{i-1}. First, denote $U = N_G(u)$ to be the neighbors of u in G, clearly , $U \subseteq \{v_2, \ldots, v_{n-1}\}$. Also, denote $N_G(w)$ to be the neighbors in G, we can infer that $N_G(w) \subseteq \{v_2, \ldots, v_{n-1}\}$. Lastly, we define $W = \{v_{i+1} \mid v_i \in N_G(w)\} \subseteq \{v_3, \ldots, v_n\}$. Notice that $|U| = \deg(u)$ and $|W| = \deg(w)$, so $|U| + |W| \geq n$ by our assumption that G satisfies the property. However, we also know that $|U \cup W| \leq n - 1$ because $v - 1 \notin U \cup W$, hence we must have $U \cap W \neq \emptyset$. This proves our claim that there exists $i \in \{3, \ldots, n-1\}$ such that $v_i \in U \cap W$, namely, $v_1 = u$ is connected to v_i and $v_n = w$ is connected to v_{i-1}.

Using our claim, we can construct a Hamiltonian cycle in G. It is really easier to see it in a diagram but here is the construction. Let P_1 be the path from u to v_{i-1} in C_H, P_2 be the path from w to v_i in C_H and denote $e_1 = \{u, v_i\}$ and $e_2 = \{w, v_{i-1}\}$. We construct $C_G = P_1 e_2 P_2 e_1$, one can verify that this is indeed a Hamiltonian path in G. □

Corollary 72. *Let $G = (V, E)$ be a graph, then $\min_{v \in V} \deg(v) \geq \frac{n}{2}$ implies that G contains a Hamiltonian cycle.*

We state this really simple corollary, because we will introduce the next section with an example to show that the bound is tight.

Example 73. Let A be an empty graph on a vertices and B be an empty graph on $a + 1$ vertices. Connect all the vertices in A to all the vertices in B with an edge. We have $\deg(v) \geq a$ for all vertices but $a < \frac{2a+1}{2}$, so we want to show that there is no Hamiltonian cycle. Indeed, any cycle must visit the same number of vertices in A and

B but it cannot do that if it visits all vertices. The kind of graph we described is the subject of the next section.

Bipartite graphs

Definition 74 (Bipartite graph)**.** We say that a graph G is **bipartite** if there exists sets $A, B \subseteq V(G)$ such that $A \cup B = V$, $A \cap B = \emptyset$ and $\forall e \in E(G), |e \cap A| = |e \cap B| = 1$.

Examples 75. Paths and cycles of even length are bipartite. Indeed, we already know that we can 2-color them, then we can have each color correspond to a set of the bipartition. It is also easy to show that trees are bipartite [9]. Moreover, subgraphs of bipartite graphs are clearly bipartite. In particular, if a graph contains a subgraph that is not bipartite, then the graph cannot be bipartite.

[9] Pick a root in the tree and consider the vertices of even distance to the root and odd distance to the root

The next theorem gives a necessary and sufficient condition for a graph to be bipartite.

Theorem 76. *A graph G is bipartite if and only if it has no odd cycle.*

We stated this theorem first because it is the result we will most likely use when trying to say if a graph is bipartite but we will prove it with an intermediary step.

Theorem 77. *Let $G = (V, E)$ be a graph, then the following are equivalent :*

1. *G is bipartite*
2. *G does not contain a closed walk of odd length*
3. *G does not contain an odd cycle*

Proof. We will denote A and B to be the sets of the bipartition of G.

(1 $\implies$ 2) Pick any closed walk $W = \{v_0, e_1, v_1, \ldots, e_m, v_m\}$. Without loss of generality, assume $v_0 = v_m \in A$. Each edge in W goes from A to B or from B to A, since the start and end of the walk are in A, there must be an even number of edges in W.

(2 $\implies$ 1) The contrapositive of this statement follows immediately from the definition because an odd cycle is a closed walk of odd length.

(3 $\implies$ 1) Without loss of generality, assume that G is connected and let T be any spanning tree of G. Let $u \in V$ and define the following bipartition for T :

$$A' = \{v \in V \mid \text{dist}_T(u, v) \equiv 0 \pmod 2\}$$
$$B' = \{v \in V \mid \text{dist}_T(u, v) \equiv 1 \pmod 2\}$$

It is left to prove that for any $e \in E(G) \setminus E(T)$, $|e \cap A'| = |e \cap B'| = 1$. Assume towards a contradiction and without loss of generality that for some $e = \{v_1, v_2\}$, we have $v_1, v_2 \in A'$. Denote C_e to be the fundamental cycle of e in T. The path P which is the cycle C_e with the edge e removed has both its endpoints in A so it must have even length. Thus, C_e is an odd cycle and we get a contradiction. □

Proposition 78. *Let G be a simple graph. G is bipartite if and only if it contains no induced cycle of odd length.*

Proof. ($\Rightarrow$) G is bipartite so it contains no odd cycle, in particular, none of its induced subgraph can be a cycle of odd length.

($\Leftarrow$) We prove the contrapositive. Suppose there is an odd cycle, then take one with shortest length, call it $C = \{v_0, e_1, \ldots, e_k, v_k\}$ where $v_0 = v_k$. If the subgraph induced by the vertices on this cycle is not a cycle, then $\exists i < j, |j - i| > 1, \{v_i, v_j\} \in E(G)$. This edge breaks C into two cycles, and one must be of odd length and smaller than C. This contradicts our choice of C. □

Matching in graphs

Definition 79 (Matching)**.** Let $G = (V, E)$ be a graph. A **matching** in G is a set of edges $M \subseteq E$ with the property that every vertex in V is incident to at most one edge in M. A matching is said to be **perfect** if all vertices are incident to exactly one edge in M.

A observation one can make fairly quickly is that for a matching M in G, we have $|M| \leq \lfloor \frac{|V|}{2} \rfloor$. Looking at this bound, one could ask how to find the largest matching in G. Before reaching for that goal, we introduce related notions.

Definition 80 (2-factor)**.** Let $G = (V, E)$ be a graph, a **2-factor** in G is a set of edges $F \subseteq E$ with the property that every vertex in V is incident to exactly two edges in F.

Example 81. If a graph contains a Hamiltonian cycle H, the set edges of H is a 2-factor in G.

Definition 82 (k-regularity)**.** A graph $G = (V, E)$ is said to be **k-regular** if $\forall v \in V, \deg(v) = k$.

Proposition 83. *For any $k \geq 1$, a (2^k)-regular graph contains a 2-factor.*

Proof. Without loss of generality, we will assume that G is connected and we will prove it by induction. For $k = 1$, every vertex is incident to exactly two edges in the graph so the set of all edges is a 2-factor. Assume that the proposition is true for some $k \geq 1$, then let G be a (2^{k+1})-regular graph which is (without loss of generality) connected.

Since the degree of all vertices is even, we know that there exists a closed Eulerian tour $T = \{v_0, e_1, \ldots, e_m, v_m\}$ where $v_0 = v_m$ and $m = |E| = 2^k n$. Since m is even, we can separate the set of edges like so :

$$E_1 = \{e_{2k} \mid k \leq \frac{m}{2}\} \qquad E_2 = \{e_{2k+1} \mid k < \frac{m}{2}\}$$

Define the subgraphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$. Observe that in the tour, each vertex is connected to the same number of edges in E_1 and E_2, so we infer that $\deg_{H_1}(v) = \deg_{H_2} = 2^k$. We can use the induction hypothesis to find two 2-factors for H_1 and H_2, clearly, the union of these 2-factors is a 2-factor for G. □

We now go back to study matchings and we will see later why the notion of 2-factors was introduced.

Definition 84 (M-alternating path)**.** Let $G = (V, E)$ be a graph and M be a matching in G. An M**-alternating path** is a path in G which alternates between edges in M and edges in $E \setminus M$. Equivalently, every internal vertex in the path is connected to his neighbors in the path by one edge in M and one edge in $E \setminus M$

Definition 85 (M-augmenting path)**.** An M**-augmenting path** is an M-alternating path of non-zero length which starts in a vertex v not incident to any edge in M and ends in a vertex $w \neq v$ not incident to any edge in M.

Lemma 86 (Berge)**.** *Let $G = (V, E)$ be a graph and M be a matching in G. M is maximum matching if and only if there is no M-augmenting paths.*

Proof. ($\Rightarrow$) Assume towards a contradiction that G contains an M-augmenting path P. The set $(E(P) \setminus M) \cup (M \setminus E(P))$ is a matching since every vertex outside the path is still matched by the same edge and every vertex in the path gets matched with the unique edge not in M and incident to it in P. It also has more edges than M since P has more edges not in M than edges in M.

($\Leftarrow$) We will prove this direction by contrapositive, namely, we will show that if M is not maximal, then there exists a M-augmenting path. We know that there exists a larger matching M', so we define $H = (V, M \cup M')$. We know that $\forall v \in V, \deg_H(v) \leq 2$, so H is the union of cycles and paths.

We have two claims that imply that there exists an M-augmenting. The first one is that all cycles in H have even length. Indeed, for each cycle C in H, each vertex must be connected to one edge in $M \setminus M'$ and one edge in $M' \setminus M$ and this cannot happen if there is an odd amount of edges. The second claim that there is a connected component with more edges in M' than M follows from the fact that $|M'| > |M|$. This component must be a path because of our first claim and this implies that this path is M-augmenting. □

In bipartite graphs, the problem of finding maximal matching is simpler and it also uses the concept of a vertex cover.

Definition 87 (Vertex cover)**.** A **vertex cover** in a graph $G = (V, E)$ is a set of vertices $X \subseteq V$ such that any edges are incident to at least one vertex in X.

Definition 88 (Sizes of minimal cover and maximal matching)**.** Let G be a graph, we will denote the size of a minimal vertex cover with $\tau(G)$ and the size of a maximal matching with $\nu(G)$.

Remark 89. We can see that $\nu(G) \leq \tau(G)$ since we need at least one vertex per edge in the maximal matching in order to cover all edges. Also, we have $\tau(G) \leq 2\nu(G)$ because the set of all endpoints of the edges in the maximal matching covers all the edges. Moreover, one can find via simple graph examples that these inequalities are tight.

Theorem 90 (Konig)**.** *Let G be a bipartite graph, then $\tau(G) = \nu(G)$.*

Proof. It is immediate that $\tau(G) \leq \nu(G)$ since for a maximum matching M, the vertex cover should contain at least one endpoint for each edge. Next, we show that $\nu(G) \leq \tau(G)$. Let A and B be the bipartition of G and M be a matching of size $\nu(G)$. Denote $A = A_M \cup A_N$ and $B = B_M \cup B_N$ where A_m and B_M are the vertices, in A and B respectively, that are matched by M and $A_N = A \setminus A_M$ and $B_N = B \setminus B_M$. Further decompose this sets in $A_M = A_X \cup A_Z$ and $B_M = B_X \cup B_Z$ with the following definitions :

$$A_Z = \{a \in A_M \mid \exists M\text{-alternating path starting in } A_N \text{ and ending in } a\}$$
$$B_Z = \{b \in B_M \mid \exists a \in A_Z, \{a, b\} \in M\}$$
$$A_X = A_M \setminus A_Z \qquad B_X = B_M \setminus B_Z$$

Next, we will make some simple observations that will lead us to find a good vertex cover.

- There is no edge in G between A_N and B_N. If it were the case, this edge could be added to M contradicting the maximality of M.
- There is no edge in G between A_N and B_X. If it were the case, we would have an M-alternating path of length two going from A_N to B_X then to A_X. This implies the vertex where this path ends in A_X should be in A_Z by definition.
- There is no edge in G between B_N and A_Z. If it were the case, we could glue this edge to the M-alternating path starting in A_N and ending in the vertex in A_Z, yielding an M-augmenting path and contradicting Berge's lemma.

- There is no edges in G between A_Z and B_X. If it were the case we could glue that edge and and edge in M from B_X to A_X to the M-alternating path starting in A_N and ending in the vertex in A_Z to form an M-alternating path from A_N to A_X, which contradicts the definition.

Figure 6. shows a graph that summarizes the observations by showing the edges in the graph that can be there. From here, we clearly see that $S = A_X \cup B_Z$ is a vertex cover. We can compute its size like so

$$|S| = |A_X| + |B_Z| = |A_X| + |A_Z| = |A_M| = |M|$$

implying that $\nu(G) \geq \tau(G)$. □

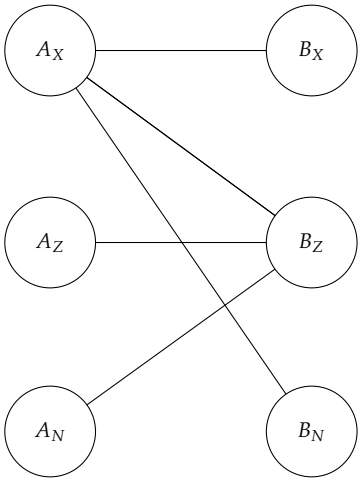

Figure 6: Summary of the observations in Konig's theorem

Definition 91 (Matching covering sets)**.** Let $A \subseteq V(G)$. An A**-covering** matching M is a matching M in G with the property that every vertex in A is incident to an edge in M.

Definition 92 (Neighbors notation)**.** Let G be a graph, $v \in V(G)$ and $S \subseteq V(G)$. We denote the set of **neighbors** of v as $N(v)$ and $N(S)$ denotes the union over $v \in S$ of $N(v)$.

Theorem 93 (Hall)**.** *Let G be a bipartite graph with bipartition A and B, then there exists an A-covering matching in G if and only if $\forall S \subseteq A, |N(S)| \geq |S|$.*

Proof. ($\Rightarrow$) Follows trivially from the pigeonhole principle.

($\Leftarrow$) Since we are looking for an A-covering matching and a vertex covering cannot be larger than one part of G. It is enough to show that $\nu(G) = |A|$ which is equivalent to $\tau(G) = |A|$ by Konig's theorem. Let X be a vertex cover in G. Denote $A' = A \setminus X$. By our hypothesis, $|N(A')| \geq |A'|$ and since $N(A') \subseteq B \cap X$, we get $|A| = |A \cap X| + |A'| \leq |A \cap X| + |B \cap X| = |X|$. Since $|A| \leq |X|$ and A is a vertex cover, we have $\nu(G) = |A|$. □

Corollary 94. *A d-regular graph bipartite contains a perfect matching.*

Proof. Let A and B be the bipartition of this graph. In order to find a perfect matching, we first need $|A| = |B|$. This is true because the number of edges leaving A and leaving B is the same, so we have $|A| \cdot d = |B| \cdot d$. Now, we only have to show Hall's condition holds for A to get an A-covering matching which will be perfect. Let $S \subseteq A$, the number of edges leaving S is less than the number of edges leaving $N(S)$ so we get $d \cdot |S| \leq d \cdot |N(S)|$ implying Hall's condition is true. □

Corollary 95. *Every $(2k)$-regular graph has a 2-factor.*

Proof. Since every vertex has an even degree, G contains an Eulerian tour. Moreover, each vertices has k incoming edges and k outgoing edges in that tour. Split up each vertex into v_i and v_o connected to the incoming edges and outgoing edges respectively. This yields a k-regular bipartite graph so it contains a perfect matching. This matching corresponds to a 2-factor in the original graph since v_i and v_o both have degree 1 in that matching. □

Corollary 96. *Every $(2k)$-regular graph has k disjoint 2-factors.*

We have enough results on bipartite graphs so let us move back to matchings in general graphs.

Definition 97 (Independent sets)**.** Let $G = (V, E)$ be a graph. A subset of vertices X is said to be **independent** if the induced subgraph $G[X]$ has no edges. We will use $\alpha(G)$ to denote the maximal size of an independent set in G.

Proposition 98. *Let $G = (V, E)$ be any graph, $\alpha(G) + \tau(G) = |V|$.*

Proof. We first show that $\alpha(G) \geq |V| - \tau(G)$. Let $C \subseteq V$ be a minimum vertex cover and let $X = V \setminus C$. X is an independent set because if there is an edge completely in X, this means C does not cover it.

Then, we show that $\tau(G) \leq |V| - \alpha(G)$. Let $X \subseteq V$ be an independent set of size $\alpha(G)$ and let $C = V \setminus X$. C is a vertex cover because if an edge has no endpoints in C, it must be completely in X contradicting its independence.

Using the two inequalities, we obtain $\alpha(G) + \tau(G) = |V|$. □

Definition 99 (Edge cover)**.** Let G be a graph without isolated points (i.e: $\forall v \in V, \deg(v) > 0$). We say that $L \subseteq E$ is an **edge cover** if every vertex is incident to at least one edge in L. We use $\rho(G)$ to denote the minimum size of an edge cover.

Proposition 100 (Gallai)**.** *Let $G = (V, E)$ be any graph, $\rho(G) + \nu(G) = |V|$.*

Proof. We first prove that $\rho(G) \leq |V| - \nu(G)$. Let M be a maximum matching in G and define $L = M \cup Z$ where Z is a set containing one incident edge for each vertex not matched by M. Note that $|Z| = |V| - 2|M|$ there are $2|M|$ vertices matched by M and each edge chosen in Z cannot be chosen twice. This gives $|L| = |M| + |Z| = |V| - |M| = |V| - \nu(G)$. Since L is an edge cover, the inequality must hold.

Then, we show that $\nu(G) \geq |V| - \rho(G)$. Let L be an edge cover of minimal size. Denote $H \subseteq G = (V, L)$. We claim that for each edge $e \in L$ there is at least one of its endpoint v with $\deg_H(v) = 1$. Assume otherwise, both endpoint have a degree higher than one

so we can remove e from L and still get a cover, contradicting the minimality of L. We can conclude that H is acyclic, or equivalently, it is a forest. Recall that $|V(H)| - |E(H)| = \text{comp}(H)$. Moreover, no component of G is of size one since L is an edge cover. Choosing one edge in each component clearly defines a matching in G of size $\text{comp}(H) = |V| - \rho(G)$.

Using the two inequalities, we obtain $\rho(G) + \nu(G) = |V|$. □

Corollary 101. *If G is bipartite, $\alpha(G) = \rho(G)$.*

We will now try to find something similar to Hall's theorem in general graphs. Namely, we would like to find a necessary and sufficient condition to find a perfect matching in G. We first observe that $|V|$ being even is already a necessary condition, but it is not sufficient. We will look at a more general condition but with the same idea behind. Before that we state a simple vocabulary definition.

Definition 102 (Odd components)**.** Let G be a graph, and $G_1, \ldots, G_k$ be its connected components. We say that some component G_i is **odd** if $|V(G_i)|$ is odd. Moreover, we use $\text{Odd}(G)$ to denote the number of odd components of G.

Theorem 103 (Tutte)**.** *Let $G = (V, E)$ be any graph, then G has a perfect matching if and only if for any subset of vertices X, $\text{Odd}(G - X) \leq |X|$.*

Proof. ($\Rightarrow$) This is the trivial implication. □

Before giving the proof of the other implication, we will explore multiple uses for it to familiarize ourself with its consequences. The first corollary is Petersen's theorem.

Theorem 104 (Petersen)**.** *All 3-regular graphs containing no cut-edges have perfect matchings.*

Corollary 105. *A 3-regular graph G has a perfect matching if and only if it has a 2-factor.*

Proof. Let M be a perfect matching in G, $G - M$ is 2-regular so $E(G) \setminus M$ is a 2-factor. Let F be a 2-factor in G, $G - F$ is 1-regular so $E(G) \setminus E(F)$ is a perfect matching. □

Proof of Petersen's theorem. Let $G = (V, E)$ be a 3-regular graph containing no cut-edges. It is enough to show Petersen's condition is satisfied. Without loss of generality, we assume that G is connected. For $X = \emptyset$, we need to check that $|V|$ is even. By the handshaking lemma, $3|V|$ is even so $|V|$ must be even. Assume towards a contradiction that $\exists X \subseteq V$ with $|X| > 0$ and $\text{Odd}(G - X) > |X|$. We claim that each odd connected component of G sends at least three edges to

X. This would imply that X has more edges than $3|X|$ contradicting the regularity of G.

In order to prove the claim, we consider multiple cases. Denote n to be the number of edges sent from some connected component C to X. Clearly, $n \geq 1$ because the initial graph G is connected. If $n = 1$, the unique edge sent must be a cut-edge contradicting our hypothesis. If $n = 2$, then, since $\forall v \in C, \deg_G(v) = 3$, we get $\sum_{v \in C} \deg_C(v) = 3|C| - 2$ which cannot be an even number, contradicting the handshaking lemma. Therefore, we can only have $n \geq 3$. □

Next, we will show that Tutte's theorem implies one direction of Hall's theorem. We will use the following observation in this proof.

Lemma 106. *Let $G = (V, E)$ be a graph with $|V|$ even. Then for any $X \subseteq V$, $\text{Odd}(G - X) \equiv |X| \mod 2$. We will refer to this as the parity observation.*

Corollary 107. *Let $G = (V, E)$ be a bipartite graph with parts A and B. Suppose that $\forall S \subseteq A, |N_G(S)| \geq |S|$, then G has an A-covering matching.*

Proof. Define $E' = \{\{b_1, b_2\} \mid b_1, b_2 \in B\}$ (i.e: all the edges between vertices in B) and define a new graph as follows:

$$G' = \begin{cases} (V, E \cup E') & |V| \text{ even} \\ (V \cup \{b'\}, E \cup E' \cup \{\{b', b\} \mid b \in B\}) & |V| \text{ odd} \end{cases}$$

This transforms the graph G by adding a new vertex to B if $|V|$ is odd and putting all the edges in B. Observe that a perfect matching in G' is an A-covering matching in G. Hence, we will check Tutte's condition, namely that for any $X \subseteq V(G')$, $\text{Odd}(G' - X) \leq |X|$. If we look at the components of $G' - X$, we see that one component has the vertices in $B' - X$[10] and the vertices in A connected to it and all the other components only have isolated vertices of A. We will call the set of these isolated vertices Y. We know that $\text{Odd}(G' - X) \leq |Y| + 1$. Now, if we show that $|X| \geq |Y|$, we will have $\text{Odd}(G' - X) \leq |X| + 1$, but by the parity observation, this is equivalent to $\text{Odd}(G' - X) \leq |X|$.

[10] $B' = B \cup \{b'\}$ if b' was added, $B' = B$ otherwise

In order to prove $|X| \geq |Y|$, we will show a tighter bound, namely, $|Y| \leq |X \cap B|$. In the original graph G, $N_G(Y) \subseteq X \cap B$ because the vertices in Y are not connected to B anymore. By Hall's condition, $|N_G(Y)| \geq |Y|$ and combining it with the last sentence, we get $|Y| \leq |X \cap B|$. □

We are now ready for the proof the second part of Tutte's theorem.

Proof of Tutte's theorem (continued). ($\Leftarrow$) We will prove it by induction on $|V| = n$. The base case is $n = 0$ and it is the implication obviously holds. Assume that it holds for every $k < n$, let $G = (V, E)$ be a graph

with $|V| = n$ and suppose that $\forall X \subseteq V, \text{Odd}(G - X) \leq |X|$, we want to show that G contains a perfect matching. We will have multiple claims that will lead us to our end goal.

Claim 1. *$|V|$ is even*

This is clear from because Tutte's condition holds with $X = \emptyset$. We will say that a set $X \subseteq V$ is Tutte-critical if $\text{Odd}(G - X) = |X|$.

Claim 2. *There exists a Tutte-critical set for G.*

The empty set is Tutte-critical. Let X_0 be the maximal Tutte-critical set (i.e: $\forall X \subseteq V$ such that $|X| \geq |X_0|$, we have $\text{Odd}(G - X) < |X_0|$).

Claim 3. *$G - X_0$ only has odd connected components.*

Suppose E is an even connected component of $G - X_0$, then take any vertex $v \in E$, the set $X_0 \cup \{v\}$ is Tutte-critical because we added one vertex to X_0 and increased the number of odd components by one. However, this contradicts the maximality of X_0. Denote $C_1, \ldots, C_k$ to be the connected components of $G - X_0$, from Claim 3, we know that they are all odd.

Claim 4. *For any $i \in \{1, \ldots, k\}$ and for any vertex $v \in C_i$, the induced subgraph $G[C_i - v]$ has a perfect matching.*

Denote $C = C_i - v$, we can use the induction hypothesis because $|C| < |G|$. Assume that $G[C]$ does not contain a perfect matching, then there exists a $Y \subseteq V(C)$ such that $\text{Odd}(C - Y) \geq |Y| + 1$. Let $X = X_0 \cup Y \cup \{v\}$, then $G - X$ still contains the odd components C_1 to C_k without C_i and at least $|Y| + 1$ more odd components (from breaking C_i). Thus, we have the following :

$$\text{Odd}(G - X) \geq k - 1 + |Y| + 1 = k + |Y| = |X_0| + |Y| \geq |X| - 1$$

By the parity observation, this gives $\text{Odd}(G - X) \geq |X|$. Since, our hypothesis says $\text{Odd}(G - X) > |X|$ is not possible, we get $\text{Odd}(G - X) = |X|$ which contradicts the maximality of X_0. From all these claims, we can conclude that if we find an X_0-covering matching with exactly one edge in each connected component of $G - X_0$, we can find a perfect matching in G.

Claim 5. *There exists a matching with the property stated above.*

Let $H = (X_0 \cup \{1, \ldots, k\}, F)$ be a graph where X_0 and $\{1, \ldots, k\}$ are a bipartition of H and F contains an edge $\{x, i\}$ if and only if $x \in X_0$, $1 \leq i \leq k$ and there exists an edge in G from X_0 to C_i. Assume that there does not exist a B-covering matching, then by Hall's theorem, $\exists Y \subseteq B, |N_H(Y)| < |Y|$ and define $X = N_H(Y)$, we have $X \subseteq X_0$ and $\text{Odd}(G - X) \geq |Y| > |X|$ contradicting Tutte's condition. Any

B-covering matching in H shows the existence of matching we are looking for in Claim 5. □

Ramsey theory

Recall the following statement from the start of the course : "Among any six people, there is always a choice of three people such that either all three people are friends or none are." We proved this result and mentioned that Ramsey theory generalizes this result. The goal here is to introduce some fundamental concepts of Ramsey theory. We first formalize the notion of coloring and give the definition of the Ramsey number.

Definition 108 (Coloring)**.** A k-coloring of the edges in K_n (the complete graph on n vertices) is a function $c : E(K_n) \to \{1, \ldots, k\}$.

If we formalize the the friendship statement, we get the following. In any 2-coloring of the edges of K_6, you can find a monochromatic triangle, namely, three edges that connect three vertices with the same color.

Remark 109. When talking about a k-coloring with k being small, we will use actual colors like red, green and blue instead of integers to facilitate our arguments.

Definition 110 (Ramsey number)**.** The Ramsey number $R(k)$ is the minimal integer n such that for any 2-coloring of $E(K_n)$, there is a subset $X \subseteq V(K_n)$ with $|X| = k$ and $K_n[X]$ has all of its edges of one color (also called monochromatic).

The friendship statement is saying that $R(3) \leq 6$ and the counter-example shown in Figure 7 shows that $R(3) > 5$, so we obtain $R(3) = 6$. One might wonder if it is always possible to find monochromatic subgraphs of arbitrary size. Ramsey's theorem answers this question.

Theorem 111 (Ramsey)**.** *For any $k \in \mathbb{N}$, $R(k) < 4^k$, implying $R(k)$ is finite.*

Before giving a proof, we will look at the known bounds for some values of k. The obvious ones are $R(1) = 1$ and $R(2) = 2$ and more generally, $R(k) \geq k$. Next, $R(3) = 6$ which we have already seen and, although it takes a lot more work to prove, we know that $R(4) = 18$. Then, we hit a wall, the next numbers are not known but we do have some bounds. For example, we know that $42 < R(5) \leq 49$ and that $101 < R(6) \leq 165$. Let us digress a bit and quote Paul Erdos.

> "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the $[R(5)]$. We could marshal

Observe that for the graph K_m with $m > R(k)$, the property will still hold because you can take any complete subgraph K_n in K_m, restrict the coloring to $E(K_n)$ and find the desired subset of $V(K_n)$ which will also be a subset of $V(K_m)$

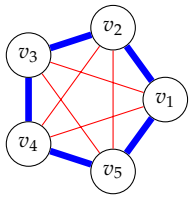

Figure 7: Counter example for K_5

> the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the $[R(6)]$, however, we would have no choice but to launch a preemptive attack."

While still postponing the proof of Ramsey's theorem, we look at a nice argument for the quadratic lower bound $R(k) > (k-1)^2$. To show this lower bound holds, we will construct a coloring on $E(K_{(k-1)^2})$ that does not have any monochromatic subgraph K_k. Divide the complete graph on $(k-1)^2$ vertices in $k-1$ groupings of $k-1$ vertices. Color the edges between the groupings in blue and the edges between the vertices of each grouping in red. Take any subgraph induced by k vertices, all the vertices cannot be in the same grouping, so all the edges are not red. Also, all the vertices cannot be in different groupings, so all the edges are not blue.

Definition 112 (General Ramsey number)**.** We will use $R(k,\ell)$ to denote the minimal integer n such that for any 2-coloring of $E(K_n)$, we can either find a red K_k or a blue K_ℓ. Note that $R(k,k) = R(k)$.

We now show the finiteness of $R(k)$.

Lemma 113.

$$R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$$

Proof. We will prove this by induction on $k+\ell$. For the base case, observe that for any k, $R(k,1) = R(1,k) = 1$ and $R(k,2) = R(2,k) = k$ since the subgraph induced by at most two vertices contains at most one edge, so it can only be monochromatic. If you cannot find two vertices connected by the color you want, then the whole graph must be of the other color so you found K_k of the other color. You can now verify the inequality holds when $k+\ell \leq 5$.

For the induction step, assume that it holds up until the sum is one less than $k+\ell \geq 6$ and fix some 2-coloring of $E(K_n)$, where $n = R(k-1,\ell) + R(k,\ell-1)$ (we know these numbers are finite by the induction hypothesis). Pick an arbitrary vertex v and observe that we have two cases. There is either at least $R(k-1,\ell)$ red neighbors of v or at least $R(k,\ell-1)$ blue neighbors of v. For the first case, we use the induction hypothesis in the red neighbors to find a blue K_ℓ or a red K_{k-1} from which we can create a red K_k by adding v. For the second case, we use the induction hypothesis in the blue neighbors to find a red K_k or a blue $K-\ell-1$ from which we can create a blue K_ℓ by adding v. □

A similar result can be used to obtain the upper bound stated in Ramsey's theorem.

Lemma 114.

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}$$

Corollary 115. $R(k) = R(k,k) < 4^k$

Proof.

$$R(k,k) \leq \binom{2k-2}{k-1} < 2^{2k-2} < 4^k$$

The second inequality is true because 2^{2k-2} is the number of subsets of a set of size $2k-2$ while $\binom{2k-2}{k-1}$ is the number of subsets of size $k-1$ of a set of size $2k-2$. □

Proof of lemma. The proof is really similar to the last lemma and uses induction. Again, we have two base cases :

$$R(1,k) = R(k,1) = 1 \leq \binom{k+1-2}{k-1} = \binom{1+k-2}{k-1}$$
$$R(2,k) = R(k,2) = k \leq \binom{k+2-2}{k-1} = \binom{2+k-2}{k-1}$$

For the induction step, assume that the inequality holds up until the sum is one less than $k+\ell \geq 6$ and fix some coloring of $E(K_n)$ where $n = \binom{k+\ell-2}{k-1}$. Pick an arbitrary vertex v, we have two cases again. There is either at least $\binom{k+\ell-3}{k-2}$ red neighbors of v or at least $\binom{k+\ell-3}{k-1}$ blue neighbors. Assume that none of these cases happen, then we obtain the following:

$$n \leq 1 + \binom{k+\ell-3}{k-2} - 1 + \binom{k+\ell-3}{k-1} - 1 = \binom{k+\ell-2}{k-1} - 1$$

leading to a contradiction. The rest of the proof is the same as for the last lemma. □

Although this upper bound can be improved, we will instead focus on improving our lower bound with a more involved argument. We will argue to obtain a value as large as we can without actually constructing the graph (like we did for the last lower bound).

Fix some $k \geq 3$ and make the following definitions.[11]

$$\mathcal{C} = \{\text{all 2-colorings of } E(K_n)\}$$
$$\mathcal{C}_Y = \{c \in \mathcal{C} \mid \exists \text{ monochromatic } K_k\}$$
$$\mathcal{C}_N = \mathcal{C} \setminus \mathcal{C}_Y$$

[11] These definitions all depend on n but we refrain from indicating in the notation to lighten the latter

We are interested in the size of these collections. Observe that the size of $\mathcal{C}$ is $2^{\binom{n}{2}}$ and that our goal is to find an n as large possible for which $|\mathcal{C}_N|$ or equivalently $|\mathcal{C}_Y| < |C|$. For a set $A \subseteq V(K_n)$ of size k and define the following collections.

$$R_A = \{c \in \mathcal{C} \mid \text{the complete subgraph induced by } A \text{ is red}\}$$
$$B_A = \{c \in \mathcal{C} \mid \text{the complete subgraph induced by } A \text{ is blue}\}$$

A simple combinatorics argument gives $|R_A| = |B_A| = 2^{\binom{n}{2}-\binom{k}{2}}$, also, we have

$$\mathcal{C}_Y = \left(\bigcup_{\substack{A \subseteq V(K_n) \\ |A|=k}} B_A \right) \cup \left(\bigcup_{\substack{A \subseteq V(K_n) \\ |A|=k}} R_A \right)$$

We will try to bound $|\mathcal{C}_Y|$ to see for what n it is smaller than $|\mathcal{C}|$. We will use what is called the union bound which can be stated as follows for any set X and Y.

$$|X \cup Y| = |X| + |Y| - |X \cap Y| \leq |X| + |Y|$$

We obtain the following derivation.

$$\begin{aligned}
|\mathcal{C}_Y| &\leq \left| \bigcup_A B_A \right| + \left| \bigcup_A R_A \right| && \text{(union bound)} \\
&= 2 \left| \bigcup_A B_A \right| && \text{(symmetry of colorings)} \\
&\leq 2 \sum_{A \in \binom{V}{k}} |B_A| && \text{(union bound)} \\
&= 2 \binom{n}{k} \cdot 2^{\binom{n}{k}-\binom{k}{2}} && \text{(counting)} \\
&= n^k \cdot 2^{\binom{n}{k}-\binom{k}{2}+1} && (\tbinom{n}{k} \leq n^k) \\
&= 2^{\binom{n}{k}-\binom{k}{2}+1+k\log_2(n)} && \text{(properties of exponent)}
\end{aligned}$$

If we want to know when $|\mathcal{C}_Y| < |\mathcal{C}|$, we need to solve for n in $2^{\binom{n}{k}-\binom{k}{2}+1+k\log_2(n)} < 2^{\binom{n}{2}}$. This is equivalent to the following.

$$\begin{aligned}
\binom{n}{2} &> \binom{n}{k} - \binom{k}{2} + 1 + k\log_2(n) \\
\binom{k}{2} &> 1 + k\log_2(n) \\
\frac{k^2-k}{2} &> 1 + k\log_2(n) \\
k &> 2\log_2(n) + \frac{2}{k} + 1
\end{aligned}$$

Choose $n = 2^{\frac{k-2}{2}}$ to obtain $k > k - 2 + 1 + \frac{2}{k}$ which holds for $k \geq 3$. This is our lower bound for $R(k)$.

Remark 116. Before further generalizing the Ramsey numbers, we want to point out that $R(k)$ can also be viewed as the minimal integer n such that any graph on n vertices contains either a complete subgraph on k vertices or an empty one (equivalently, $\alpha(G) \geq k$).

Definition 117 (More general Ramsey number)**.** We use $R_\ell(k)$ to denote the minimal integer n such that for any ℓ-coloring of $E(K_n)$, we can find a monochromatic subgraph induced by k vertices. We can also use $R_\ell(k_1, \ldots, k_\ell)$ to denote the minimal integer n such that for any ℓ-coloring of $E(K_n)$, we can either find a K_{k_1} of color k_1 or a K_{k_2} of color k_2, etc...

Theorem 118. *For any ℓ and k, $R_\ell(k)$ is finite.*

Proof. We prove it by induction, for $\ell = 1, 2$ the proof is already done. Assume that the theorem holds up to some ℓ, we claim that

$$R_{\ell+1}(k) = R_{\ell+1}(k, \overset{\ell+1}{\ldots}, k) \leq R_\ell(k, \overset{\ell-1}{\ldots}, k, R_2(k, k))$$

Take any complete graph of size bigger than the R.H.S. and any $(\ell+1)$-coloring c. Reduce this coloring to an ℓ-coloring by combining the colors ℓ and $\ell+1$ together. If we can find a monochromatic K_k of color in $\{1, \ldots, \ell-1\}$, we are done because it will be also monochromatic with the coloring c. If we cannot find such coloring, then we must be able to find a monochromatic $K_{R_2(k)}$. If we look at this subgraph with the coloring c, it has only two colors so it must contain a monochromatic K_k of color ℓ or $\ell+1$. □

We now use the Ramsey numbers in a number theoretic puzzle.

Question 119. *Can you color the natural numbers*[12] *with red and blue such that there is no monochromatic solution to $x + y = z$?*

[12] We are working with $\mathbb{N}^+$ since $0 + 0 = 0$ would always be a monochromatic solution

By a monochromatic solution, we mean that $c(x) = c(y) = c(z)$ and $x + y = z$. Let us start coloring and see what happens. By symmetry, we can arbitrary choose 1 to be red. Since $1 + 1 = 2$, 2 must be blue to have a valid coloring. Since $2 + 2 = 4$, 4 must be colored red and then $1 + 4 = 5$ means 5 must be blue. Then, 3 can be either red or blue. In the first case, $1 + 3 = 4$ is a monochromatic solution. In the second case, $2 + 3 = 5$ is a monochromatic solution.

In fact, it is possible to generalize this result to any numbers of colors.

Theorem 120 (Schur)**.** *For any $\ell \in \mathbb{N}$, $\mathbb{N}^+$ is not ℓ-colorable such that $x + y = z$ has no monochromatic solution.*

Proof. We will prove an equivalent statement : for any $\ell \in \mathbb{N}$, there exists an integer $n(\ell)$ such that any ℓ-coloring of $\{1, \ldots, n(\ell)\}$ has a monochromatic solution to that equation.

We claim that $n(\ell) = R_\ell(3) - 1$. Denote c to be an arbitrary ℓ-coloring of $\{1, \ldots, n(\ell)\}$. Take the complete graph on the integers $\{1, \ldots, R_\ell(3)\}$ and consider the following ℓ-coloring d of $E(K_{R_\ell(3)})$: $\forall i < j, d(\{i, j\}) = c(j - i)$. Note that $1 \leq j - i \leq R_\ell(3) - 1$ so this is

well defined. By the definition of $R_\ell(3)$, there exists a monochromatic triangle in d. Namely, there exists integers $x < y < z$ such that $d(\{x,y\}) = d(\{y,z\}) = d(\{x,z\})$. However, by how we defined d, this is equivalent to $c(y-x) = c(z-y) = c(z-x)$. We get a monochromatic solution since $y - x + z - y = z - x$. □

Connectivity of graphs

This section will introduce the notion of connectivity. This part of graph theory arose during the war, when Russian were trying to figure out how to disconnect the railway network by destroying the least amount of connections. Another problematic was to find the maximal amount of resources could be sent through the same network. As it turns out these two problems lead to the same theory.

There are two ways to define connectivity in a graph, using the vertices or the edges. We will study those in parallel as the results will be similar but not always the same.

Definition 121 (Vertex-disjoint paths)**.** Let $G = (V, E)$ be a simple graph and $u \neq w \in V$. We will use $P_G(u,w)$ to denote the maximal size of a collection of internally vertex-disjoint paths from u to w. By internally vertex-disjoint, we imply that all vertices except for u and w are in at most one path of the collection.

Clearly, $P_G(u,w)$ is positive and $P_G(u,w) = 0$ if and only if $u \not\sim w$. Moreover, since paths cannot contain the same vertices and at most one path contains only 2 vertices, we have $P_G(u,w) \leq |V| - 1$. We define the connectivity of G as follows :

$$\kappa(G) = \min_{u \neq w \in V} P_G(u,w)$$

One can remark that $\kappa(G) = 0$ if and only if G is disconnected.

Definition 122 (Edge-disjoint paths)**.** Let $G = (V, E)$ be a multigraph and $u \neq w \in V$. We will use $P'_G(u,w)$ to denote the maximal size of a collection of edge-disjoint paths. By edge-disjoint, we imply that all edges are in at most one path of the collection.

Once again, $P'_G(u,w)$ is positive and zero if and only if $u \not\sim w$. However, since we are working in a multigraph, it is impossible to give an upper bound. Indeed, the graph could have an arbitrary number of edges from u to w and $P'_G(u,w)$ would be at least this number. The new definition of connectivity we give is the following :

$$\kappa'(G) = \min_{u \neq w \in V} P'_G(u,w)$$

Examples 123.
- For an empty graph on a single vertex, we have $\kappa(G) = \kappa'(G) = 1$.

- For a complete graph, we have $\kappa(K_n) = \kappa'(K_n) = n - 1$.
- For a tree T, we have $\kappa(T) = \kappa'(T) = 1$ since all there are unique paths between any vertices.
- For a cycle, we have $\kappa(C_n) = \kappa'(C_n) = 2$ since you can only go from a vertex to another by following the cycle in one orientation or the other.
- These two values are not always the same. For a graph composed of two cliques[13] of size n_1 and n_2 with 2 vertices in common, we have $\kappa(G) = 2$ and $\kappa'(G) = \min(n_1 - 1, n_2 - 1)$.

[13] A clique is a synonym for a complete subgraph

Definition 124 (Vertex cut)**.** A subset $C \subseteq V$ is a vertex cut if $G[V \setminus C]$ is disconnected. $\{v\}$ is a vertex cut if and only if v is a cut vertex.

For any vertex cut C, $\kappa(G) \leq |C|$ because if $G[V \setminus C]$ has two connected components A and B, the vertices $u \in A$ and $w \in B$ have at most $|C|$ vertex disjoint paths between them, otherwise they would still be connected.

Theorem 125 (Menger)**.** *Let G be a simple and not complete graph, then*

$$\kappa(G) = \min_{C \text{ vertex cut}} |C|$$

In the literature, you might find this to be the definition of connectivity

Definition 126 (Edge cut)**.** A subset $F \subseteq E$ is an edge cut if $G' = (V, E \setminus F)$ is disconnected. $\{e\}$ is an edge cut if and only if e is a cut edge.

For any edge cut F, $\kappa'(G) \leq |F|$ because if G' has two connected components A and B, the vertices $u \in A$ and $v \in B$ have at most $|F|$ edge disjoint paths between them, otherwise they would still be connected.

Theorem 127 (Ford-Fulkerson)**.** *Let G be a multigraph, then*

$$\kappa'(G) = \min_{F \text{ edge cut}} |F|$$

In order to prove these theorems, we will use a prove two lemmas, one can view these as the "local" versions of the theorems.

Definition 128. Let $G = (V, E)$ be a graph and $u \neq w \in V$ such that $\{u, w\} \notin E$. We will use $c_G(u, w)$ to denote the minimum size of a vertex cut C not containing u nor w such that C disconnects u and w.

Lemma 129. *Let $G = (V, E)$ be a simple graph. For any $u \neq w \in V$ such that $\{u, w\} \notin E$, we have $c_G(u, w) = P_G(u, w)$.*

Proof. We will use induction on $|E|$, the base case $|E| = 0$ is trivial. For the induction step, assume for all graphs with less edges than G and let $u \neq w \in V$ be two non-adjacent vertices. Also, denote $k = c_G(u, w)$. It is clear that $P_G(u, w) \leq k$ because if you remove k vertices in those disjoint paths, u and w must become disconnected. The other inequality is less trivial and involves two cases.

Suppose that every edge in E has an endpoint in $\{u, w\}$. The minimal vertex cut is the only vertex cut separating u and w, namely, $C = N_G(u) \cap N_G(w)$. This is true because removing these vertices will disconnect u and w and any other subset of V will either contain u or w or it will not contain some vertex in C, leaving u and w connected via that vertex. Also, we clearly have $P_G(u, w) = |C| = k = c_G(u, w)$ and we are done.

In the second case, there exists some edge $f = \{a, b\}$ where $f \cap \{u, w\} = \emptyset$. Define a new graph $H = G - f$ and observe that $P_H(u, w) \leq P_G(u, w)$ since all the paths in H are paths in G. Also, we have $c_G(u, w) \leq c_H(u, w) + 1$ since if C is a cut disconnecting u and w in H, then $C \cup \{a\}$ must disconnect u and w as well. We can rewrite this inequality as $c_H(u, w) \geq k - 1$ and our induction hypothesis gives $P_H(u, w) = c_H(u, w) \geq k - 1$ because H has less edges than G. If both of these are actually equal to k, then we must have $P_G(u, w) = k$ and we are done. Assume otherwise and denote $C = \{v_1, \ldots, v_{k-1}\}$ be a vertex cut disconnecting u and w.

Going back to the graph G, we know that C does not disconnect u and w because $|C| < c_G(u, w) = k$. This implies that there is a path from u to w passing through the edge f. Without loss of generality, say that $u, a \in U$ and $w, b \in W$ where U and W are components of $H - C$. Define the following graphs :

$$
\begin{aligned}
G_u &= ((V \setminus U) \cup \{\bar{u}\}, E_u)) \\
&\quad \text{where } E_u = \{e \in E \mid e \cap U = \emptyset\} \cup \{\{\bar{u}, y\} \mid x \in U, \{x, y\} \in E\} \\
G_w &= ((V \setminus W) \cup \{\bar{w}\}, E_w)) \\
&\quad \text{where } E_w = \{e \in E \mid e \cap W = \emptyset\} \cup \{\{\bar{w}, y\} \mid x \in W, \{x, y\} \in E\}
\end{aligned}
$$

Note that any vertex cut in G_u that separates $\bar{u}$ and w will separate u and w in G as well. Hence, $c_{G_u}(u, w) \geq k$. On the other hand, the vertex cut $\overline{C} = C \cup \{b\}$ separates $\bar{u}$ and w in G_u so $c_{G_u}(\bar{u}, w) = k$.

We can use the induction hypothesis on G_u because all edges coming out U are represented by at most one edge per vertex not in U and at least one edge inside U is removed because $u, a \in U$. Therefore, we have k internally disjoint paths from $\bar{u}$ to w, denoted $P_1, \ldots, P_k$. With a symmetric argument for G_w, we obtain a collection $Q_1, \ldots, Q_k$ of internally disjoint paths from $\bar{w}$ to u. Each paths in each collection must go through $\overline{C}$ and they are disjoint, so they pass by

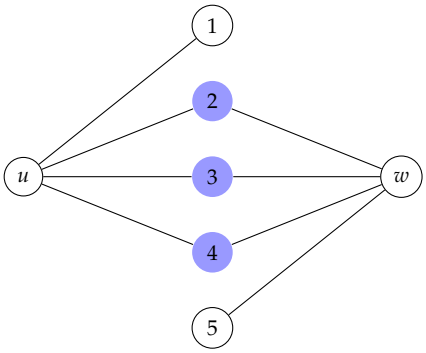

Figure 8: This is an example of a graph for the first case, the highlighted nodes are in the intersection of the neighborhoods and form the only vertex cut separating u and w

exactly one vertex in $\overline{C}$. Without loss of generality, say that P_i and Q_i pass through v_i for $i \in \{1, \dots, k-1\}$ and say P_k and Q_k pass through a and b respectively.

We will construct k internally disjoint paths in G. For $i \in \{1, \dots, k-1\}$, the path starting at u, following Q_i until v_i and then following P_i until w will be internally disjoint from the path with the same construction and $j \neq i$. This gives $k-1$ paths. The last one starts at u and follows Q_k until a goes to b with the edge f and follows P_k until w. □

We use this lemma to prove Menger's theorem.

Proof of Menger. Let $G = (V, E)$ be simple and not complete, so there exists some vertices which are not adjacent. Pick $u \neq w \in V$ such that $P_G(u, w)$ is minimal and denote $\kappa(G) = k$. If u and w are not adjacent, then using the lemma, $c_G(u, w) = P_G(u, w) = k$.

Suppose that u and w are adjacent and consider the graph $H = (V, E \setminus \{u, w\})$. We clearly have $P_H(u, w) = P_G(u, w) - 1 = k - 1$ and from the lemma, we obtain $c_H(u, w) = k - 1$. Let C be a vertex cut of size $k-1$ separating u and w. Assume towards a contradiction that $C = V \setminus \{u, w\}$, then $|C| = |V| - 2$ implying that $k = |V| - 1$ and we have seen that this implies G is complete, leading to a contradiction.

We can now pick some vertex $x \in V(H) \setminus C$ with $x \neq u$ and $x \neq w$. Without loss of generality, in $H - C$, x is not connected to w, implying that $C \cup \{u\}$ separates x and w. We get $c_G(x, w) \leq k$, so we found a vertex cut of size k. □

Definition 130. Let $G = (V, E)$ be a graph and $u \neq w \in V$. We will use $c'_G(u, w)$ to denote the minimum size of an edge cut F such that F disconnects u and w.

Lemma 131. *Let $G = (V, E)$ be a multigraph. For any $u \neq w \in V$, we have $c'_G(u, w) = P'_G(u, w)$.*

We can view this lemma in a different way which somewhat links this to the next section.

Definition 132 (Boundary)**.** Let $X \subseteq V(G)$, we define the boundary of X (denote it $\partial(X)$) as the set of edges touching exactly one vertex of X, namely, $\partial(X) = \{e \in E \mid |e \cap X| = 1\}$.

Remark 133. Observe that

$$c'_G(u, w) = \min_{\{u\} \subseteq X \subseteq V \setminus \{w\}} |\partial_G(X)| =$$

In order to prove this result and Ford-Fulkerson's theorem, we will develop a more general theory about networks and flows.

Networks

As usual when starting a new section, we will start with a series of definitions and observations to get familiar with the objects we will work with.

Definition 134 (Digraph)**.** A digraph D is a tuple (V, E) where $E \subseteq V \times V$. The edges are now ordered pairs and contain information about the orientation of the edge. It is different from an oriented graph since it could have two edges that go from one vertex to another and back, namely, for $v, w \in V$, $(v, w), (w, v) \in E$.

Definition 135 (Directed path)**.** A directed path in D is a path $P = \{v_0, e_1, v_1, \ldots, e_k, v_k\}$ such that $e_i = (v_{i-1}, v_i)$. P goes from v_0 to v_k and not the opposite.

Definition 136 (Oriented boundaries)**.** Let $X \subseteq V$, we define the oriented boundaries of X:

$$\partial^+(X) = \{(u, w) \in E \mid u \in X, w \notin X\} = \text{outgoing edges}$$
$$\partial^-(X) = \{(u, w) \in E \mid u \notin X, w \in X\} = \text{incoming edges}$$

Lemma 137. *Let $D = (V, E)$ be a digraph and $s \neq t \in V$, then either there exists a directed path from s to t or there exists a subset $X \subseteq V$ with $\{s\} \subseteq X \subseteq V \setminus \{t\}$ such that $\partial^+(X) = \emptyset$.*

Proof. Let $X = \{x \in V \mid \exists \text{ directed path from } s \text{ to } x\}$. Clearly $\partial^+(X) = \emptyset$ as if it were not empty, there would be a directed path from s to the head of an edge in the boundary, so the head must be in X which is a contradiction. Also, $s \in X$ by the empty path. Now, either $t \in X$ and there exists a directed path from s to t or $t \notin X$ and X is the desired subset. □

Definition 138 (Flow)**.** Let $D = (V, E)$ be a digraph and $s \neq t \in V$. A map $\phi : E \to \mathbb{R}_0^+$ is an s,t-flow if $\forall x \in V \setminus \{s, t\}$, the following holds[14]:

$$\sum_{e \in \partial^+(x)} \phi(e) = \sum_{e \in \partial^-(x)} \phi(e)$$

[14] This is often called Kirchoff's law, you might recognize it from a course in electromagnetism

We define the value of an s,t-flow as follows:

$$\text{val}(\phi) = \sum_{e \in \partial^+(s)} \phi(e) - \sum_{e \in \partial^-(s)} \phi(e) = \sum_{e \in \partial^-(t)} \phi(e) - \sum_{e \in \partial^+(t)} \phi(e)$$

Lemma 139. *Let $D = (V, E)$ be a digraph, $s \neq t$ be vertices and ϕ be an s,t-flow of value k, then $\forall \{s\} \subseteq X \subseteq V \setminus \{t\}$, we have*

$$\sum_{e \in \partial^+(X)} \phi(e) - \sum_{e \in \partial^-(X)} \phi(e) = k$$

Proof. Define the following subset of edges:

$$\begin{aligned} E_0 &= \{e \in E \mid |e \cap X| = 0\} \\ E_1 &= \{e \in E \mid |e \cap X| = 2\} \\ E_H &= \{(u,w) \in E \mid u \notin X, w \in X\} \\ E_T &= \{(u,w) \in E \mid w \notin X, u \in X\} \end{aligned}$$

It is obvious that $E = E_0 \cup E_1 \cup E_H \cup E_T$ and that all this sets are disjoint. Also, note that $E_H = \partial^-(X)$ and $E_T = \partial^+(X)$.

$$\begin{aligned} k &= \sum_{e \in \partial^+(s)} \phi(e) - \sum_{e \in \partial^-(s)} \phi(e) \\ &= \sum_{x \in X} \left(\sum_{e \in \partial^+(x)} \phi(e) - \sum_{e \in \partial^-(x)} \phi(e) \right) \\ &= \sum_{e \in E_0} 0 + \sum_{e \in E_1} 0 + \sum_{e \in E_H} -\phi(e) + \sum_{e \in E_T} \phi(e) \\ &= \sum_{e \in \partial^+(X)} \phi(e) - \sum_{e \in \partial^-(X)} \phi(e) \end{aligned}$$

□

Definition 140 (Capacity function)**.** Let $D = (V, E)$ be a digraph, a capacity function is a map $c : E \to \mathbb{R}_0^+$.

Definition 141 (Network)**.** A network is a 5-tuple (V, E, s, t, c) where (V, E) is a digraph, $s \neq t \in V$ are called the source and target(or sink) of the network and c is a capacity function.

Definition 142 (c-admissibility)**.** An s,t-flow is c-admissible if $\forall e \in E, \phi(e) \leq c(e)$.

Remark 143. Although all of the following theory will work when the range of ϕ is the positive real numbers, there is some theory about the maximum of functions in multidimensional spaces that needs to be used and we do not want to mention it in this class. Therefore, we will assume that ϕ is an integral function.

Lemma 144. *Let $D = (V, E)$ be a digraph, $s \neq t$ be vertices and ϕ be an integral s,t-flow of value k, then there exists a collection of paths $\{P_1, \ldots, P_k\}$ all going from s to t such that every edge $e \in E$ belongs to at most $\phi(e)$ paths.*

Proof. We prove it by induction on k. When $k = 0$, the property is vacuously true.

Suppose the property holds for $k-1$, with $k \geq 1$. Define $D' = (V, \{e \in E, \phi(e) > 0\})$. If there is no directed path from s to t in D', by lemma 135, there exists a subset $\{s\} \subseteq X \subseteq V \setminus \{t\}$ such that

$\partial^+(X) = \emptyset$. However, by lemma 137 and the fact that ϕ is positive, we get that $k < 0$ which contradicts our beginning assumption.

Let P_k be a directed path form s to t in D' and define the map $\psi : E \to \mathbb{N}_0$ as follows:

$$\psi(e) = \begin{cases} \phi(e) & e \notin P_k \\ \phi(e) - 1 & e \in P_k \end{cases}$$

Kirchoff's law still holds for every internal vertex in the path since we remove 1 from an outgoing edge and 1 from an incoming edge. Hence, this map is an s,t-flow in D. Moreover, $\text{val}(\psi) = k - 1$ since we remove 1 from exactly one outgoing edge of s. We can now use the induction hypothesis to find paths $P_1, \ldots, P_{k-1}$ such that every edge e in D is in at most $k-1$ paths. Adding the last path P_k still maintains the desired property. □

Definition 145 (Augmenting path)**.** An undirected path $P = \{v_0, e_1, \ldots, e_\ell, v_\ell\}$ from s to t in a network (V, E, s, t, c) is called augmenting for an s,t-flow ϕ if $\forall i \in \{1, \ldots, \ell\}, e_i = (v_i, v_{i+1}) \implies \phi(e) \leq c(e) - 1$ and $e_i = (v_{i+1}, v_i) \implies \phi(e) \geq 1$.

Lemma 146. *Let (V, E, s, t, c) be a network, ϕ be an integral c-admissible s,t-flow and P be an augmenting path for ϕ. Then there exists a c-admissible s,t-flow ψ with* $\text{val}(\psi) \geq \text{val}(\phi) + 1$.

Proof. Define the following map :

$$\psi(e) = \begin{cases} \phi(e) + 1 & \text{if } P \text{ uses } e \text{ in the forward direction} \\ \phi(e) - 1 & \text{if } P \text{ uses } e \text{ in the backward direction} \\ \phi(e) & \text{otherwise} \end{cases}$$

Observe that $\forall e \in E, 0 \leq \psi(e) \leq c(e)$ by the definition of an augmenting path and the map ψ. Let $x \in V \setminus \{s, t\}$ inside the path P. Say that x appears in the path in the following sequence $vexfw$ where $v, w \in V$ and $e, f \in E$. We have four cases.

- $e = (v, x)$ and $f = (x, w)$ yields $\psi(e) = \phi(e) + 1$ and $\psi(f) = \phi(f) + 1$
- $e = (v, x)$ and $f = (w, x)$ yields $\psi(e) = \phi(e) + 1$ and $\psi(f) = \phi(f) - 1$
- $e = (x, v)$ and $f = (x, w)$ yields $\psi(e) = \phi(e) - 1$ and $\psi(f) = \phi(f) + 1$
- $e = (x, v)$ and $f = (w, x)$ yields $\psi(e) = \phi(e) - 1$ and $\psi(f) = \phi(f) - 1$

All these four cases maintain Kirchoff's property for x showing that ψ is an s,t-flow. Moreover, $\text{val}(\psi) = \text{val}(\phi) + 1$ since we either add 1 to the flow of an outgoing edge or we remove 1 from the flow of an incoming edge. This finishes the proof of the lemma. □

We are now going to show the main result of this section. This is the network version of the Ford-Fulkerson theorem we have introduced in the last section.

Definition 147 (Capacity of a set)**.** Let (V, E, s, t, c) be a network and $\{s\} \subseteq X \subseteq V \setminus \{t\}$. The capacity of the set is denoted $\text{cap}(X)$ and defined

$$\text{cap}(X) = \sum_{e \in \partial^+(X)} c(e)$$

Theorem 148 (Ford-Fulkerson)**.** *Let (V, E, s, t, c) be a network and Φ be the set of all c-admissible s,t-flows, then, we have the following:*

$$\max_{\phi \in \Phi} \text{val}(\phi) = \min_{\{s\} \subseteq X \subseteq V \setminus \{t\}} \text{cap}(X)$$

Proof. For the $\leq$ side, let $\{s\} \subseteq X \subseteq V \setminus \{t\}$ be an s,t-cut of minimum capacity k. Clearly, any flow needs to pass through X so it cannot have a value more than k.

For the $\geq$ side, let ϕ be an s,t-flow of maximum value k. Let $X = \{x \in V \mid \exists \phi\text{-augmenting path from } s \text{ to } x\}$. By Lemma 144, we know that there cannot be an augmenting path for ϕ so $t \notin X$. Also, $s \in X$ by the empty path. Moreover, we have that $\forall e \in \partial^+(X), \phi(e) = c(e)$ and $\forall e \in \partial^-(X), \phi(e) = 0$, otherwise we would e would extend an augmenting path outside X. Therefore, we use Lemma 137 to obtain

$$k = \sum_{e \in \partial^+(X)} c(e) = \text{cap}(X)$$

This shows that $k \geq \min_{\{v\} \subseteq X \subseteq V \setminus \{t\}} \text{cap}(X)$. □

In the next section, we go back to undirected graphs.

Proper Vertex Coloring

Recall that a graph G is called bipartite if $V(G) = R \cup B$ such that $G[R]$ and $G[B]$ are empty. In this section, we try to generalize this notion.

Definition 149 (Proper vertex coloring)**.** Let $G = (V, E)$ be a simple graph. A proper k-vertex coloring of G is a map $c : V \to \{1, \ldots, k\}$ such that $\forall v \neq w \in V, c(v) \neq c(w)$, namely there is no monochromatic edge. The chromatic number of G, denoted $\chi(G)$, is the minimum k such that there exists a proper k-vertex coloring of G.

Examples 150.

- If B is a bipartite graph with at least one edge, then $\chi(B) = 2$.
- If G has no edges, $\chi(G) = 1$.
- For an even cycle, we have $\chi(C_{2\ell}) = 2$. For an odd cycle, we have $\chi(C_{2\ell+1}) = 3$
- $\chi(K_n) = n$

Proposition 151. *Let G be a simple graph, recall that $\alpha(G)$ is the size of the largest independent set. We have $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

Proof. Let c be a proper vertex k-coloring and denote $V = X_1 \cup \cdots \cup X_k$ where $X_i = \{x \in V \mid c(x) = i\}$. Clearly, each set X_i is independent, so we have the following:

$$|V| = \sum_{i=1}^{k} |X_i| \leq \alpha(G) \cdot k \leq \alpha(G) \cdot \chi(G)$$

The inequality follows. □

Remark 152. If $H \subseteq G$ and $\chi(H) \geq k$, then $\chi(G) \geq k$.

We might ask what makes the chromatic number of G bigger. One might notice that large cliques imply a large chromatic number and wonder if the converse is true. In fact, it is not and the following theorem shows that in a really strong sense.

Definition 153 (Clique number)**.** For a graph G, we define the clique number of G, and denote $\omega(G)$, to be the maximum number ℓ such that $K_\ell \subseteq G$.

It is easy to see that $\chi(G) \geq \omega(G)$, but that bound is not that tight since we can construct a triangle free graph with chromatic number k, for any k.

Theorem 154. *For any $k \in \mathbb{N}$, there exists a graph G_k simple graph with no triangles and $\chi(G_k) > k$.*

Proof. Fix some $k \in \mathbb{N}$. Let $n = R_k(3)$ and $N = \binom{n}{2}$, it is the number of edges in K_n. Let $G_k = (V, E)$ where V contains all the edges K_n and E contains all the edges $\{\{a, b\}, \{c, d\}\}$ with $a < b = c < d$. We first claim that G_k is triangle-free. Suppose that some vertices $\{a, b\}$, $\{c, d\}$ and $\{e, f\}$ form a triangle. Without loss of generality, we have the following two options :

$$a < b = c < d = e < f = a$$
$$a < b = c < d = e < f = b$$

Both lead to a contradiction. The second claim is that $\chi(G_k) > k$, i.e: for any map $c : V \to \{1, \ldots, n\}$, there exists $e \in E$ such that both ends of e have the same color in c. Fix some c and observe that it is already a coloring of the edges of K_n because these edges are the vertices of G_k. By the definition of the Ramsey number $R_k(3)$, there exists a monochromatic triangle in K_n. Namely, there exists $x < y < z \in \{1, \ldots, n\}$ such that $c(\{x, y\}) = c(\{y, z\})$. This shows that c is not a proper coloring. □

Theorem 155. *Let $G = (V, E)$ be a graph without triangles with $n = |V|$, then $\chi(G) \leq \sqrt{2n}$.*

Proof. Without loss of generality, we assume that G is connected and prove it by induction on n. For the base case $n = 1$, we have $\chi(G) = 1 \leq \sqrt{2}$. Suppose it holds for values smaller than n, with $n > 1$, we consider two cases.

If $\forall v \in V, \deg(v) \leq \lfloor \sqrt{2n} \rfloor =: k$, by Brooks' theorem, we have $\chi(G) \leq \lfloor \sqrt{2n} \rfloor$ unless $G = K_n$ or $G = C_n$ when n is odd. For the complete graph, n must be 2 since otherwise G would not be triangle-free. If $G = K_2$, we have $\chi(K_2) = 2 \leq \sqrt{4}$. For the odd cycle, n must be at least 5 since C_3 has a triangle. If $G = C_n$ with $n \geq 5$ an odd integer, we have $\chi(G) = 3 \leq \sqrt{2n}$.

If $\exists v \in V$ with $\deg(v) \geq k + 1 \geq \lceil \sqrt{2n} \rceil$. We have that $N(v)$ is an independent set of size at least $k + 1$. Denote $G' = G - N(v)$. By induction, $\chi(G') \leq \sqrt{2(n - \lceil \sqrt{2n} \rceil)}$, implying that $\chi(G) \leq \sqrt{2(n - \lceil \sqrt{2n} \rceil)} + 1$. Since you would need just one color for $N(v)$. Now, our aim is to show $\sqrt{2(n - \lceil \sqrt{2n} \rceil)} + 1 \leq \sqrt{2n}$. We have the following equivalences.

$$\begin{aligned}
\sqrt{2(n - \lceil \sqrt{2n} \rceil)} + 1 &\leq \sqrt{2n} \\
\sqrt{2(n - \sqrt{2n})} &\leq \sqrt{2n} - 1 \\
2n - 2\sqrt{2n} &\leq 2n - 2\sqrt{2n} + 1 \quad \text{(by squaring both sides)}
\end{aligned}$$

□

Definition 156 (Maximum degree)**.** We will denote $\Delta(G)$ to be the maximum degree of a vertex in G.

Another easy observation is that $\chi(g) \leq \Delta(G) + 1$, but we can make this bound tighter with the following theorem.

Theorem 157 (Brooks)**.** *Let G be a connected loopless multigraph that is not complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. Pretty long proof that we might not do in class. □

Definition 158 (d-degeneracy)**.** A graph G is called d-degenerate, if $\exists v \in \deg(v) \leq d$ and $G - v$ is also degenerate. Equivalently, every subgraph of G has a vertex of degree less or equal to d.

Proposition 159. *If G is d-degenerate, then $\chi(G) \leq d + 1$.*

Proof. We use induction on $n = |V|$. If $n = 1$, this is trivial. Now, assume it holds up to $n - 1$. Then take a vertex $v \in V, \deg(v) \leq d$, by induction, you can color $G - v$ with $d + 1$ colors since $G - v$ is also d-degenerate. Now, since v has at most d neighbors, there is one color that was not used in $N(v)$. Thus, we can finish the coloring of G with that color. □

Definition 160. A proper k-edge-coloring of a loopless multigraph $G = (V, E)$ is a map $c : E \to \{1, \ldots, k\}$ such that $\forall e, f \in E, c(e) \neq c(f)$. Namely, we can decompose $E = E_1 \cup \cdots E_k$ where each E_i is a matching. $\chi'(G)$ is the minimum k such that there exists a k-edge coloring for G.

Examples 161.

- $\chi'(C_{2\ell}) = 2$
- $\chi'(C_{2\ell+1}) = 3$
- $\chi'(G) \geq \max_{v \in V(G)} \deg(v)$

Theorem 162 (Vizing)**.** *If G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$. If G is not simple, then denote $\mu(G)$ to be the maximum multiplicity of an edge in G, we have $\chi'(G) \leq \Delta(G) + \mu(G)$.*

Proof. Later... □

Theorem 163 (Konig's line coloring)**.** *If $G = (V, E)$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Proof. We show this by induction on $n = |E|$. When $|E| = 1$, it is obvious that $\chi'(G) = 1 = \Delta(G)$. Assume this holds up to $n - 1$ and we have a bipartite graph G with $|E| = n$. Pick some edge $e = \{u, w\} \in E$ and define $G' = G - e$. Clearly, G' is bipartite and $\max_{v \in V(G')} \deg_{G'}(v) \leq \Delta(G)$. Hence, by our induction hypothesis, there exists a proper edge coloring of G' using $\Delta(G)$ colors. Also, note that $\deg_{G'}(u), \deg_{G'}(w) \leq \Delta(G) - 1$, namely, there exists colors α_u and α_w such that no edge of those colors are incident to u and w respectively. If $\alpha_u = \alpha_w$, extend the coloring of G' to a coloring of G by coloring e with α_u and we are done.

Suppose that $\alpha_u \neq \alpha_w$. Consider the subgraph of G' with all the edges colored with α_u or α_w and call it H. We know that this graph is composed of cycles and paths and that $\deg_H(u) = \deg_H(w) = 1$.

Therefore, u and w can only be the ends of paths in H. We have two cases.

If u and w are not in the same connected components, then, on the path of u, change all the edges of color α_u to α_w and vice-versa. Now, no edge of color α_w is incident to u and we can color e with α_w to get a proper edge coloring of G. If u and w are on the same path P, they are the ends of that path. P must be of even length because the edges must alternate color and the two end edges are of different colors. However, this implies that $P + e$ is an odd cycle contradicting the fact that G is bipartite. □

Theorem 164 (Shannon)**.** *If G is a loopless multigraph, $\chi'(G) \leq 3\lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. Let $k = \lceil \frac{\Delta(G)}{2} \rceil$, there exists a graph $H \supseteq G$, that is $2k$-regular[15]. H splits into k 2-factors. We can color each of these with 3 colors, so we can color H (hence G) with $3k = 3\lceil \frac{\Delta(G)}{2} \rceil$ colors. □

[15] Its construction is easy, create a copy G and connect parallel edges from vertices v to their copy until they have degree $2k$

Structural graph theory

This section will mostly review properties of planar graphs. There will be parts where we do not use lots of formalism since introducing it requires topology notions that are out of the scope of this course.

Definition 165 (Planar graph)**.** A graph G is planar if it has a drawing in the plane $\mathbb{R}^2$ such the edges do not intersect themselves or other edges except at the vertices (which are points on the plane).

Examples 166.

- Trees are planar graphs. Pick a root arbitrarily, draw it with its neighbors such that this part of the graph is planar. Now, zoom in on each neighbor, considering it as the root of the subtree it is the ancestor to. Draw the subtrees recursively.
- Cycles are planar graphs. Draw a circle and put as many points on it as there are vertices in the graph.
- Empty graphs are planar. Just draw the vertices on different points of the plane.
- What about complete graphs. We give planar drawings for K_2, K_3 and K_4, but we will see that K_5 is not planar.

  That last example raises the following question.

Question 167. *How can we show that a graph is not planar ?*

We quickly see that proofs by exhaustion are not possible here. Indeed, although graphs are finite, their drawing is on $\mathbb{R}^2$, so there

is a continuum of possibilities to draw it. Here is a more formal definition of a drawing.

Definition 168 (Drawing)**.** Let $G = (V, E)$ be a graph. A drawing or plane graph D for G in $\mathbb{R}^2$ is the following list of functions. First, there is an injective (so that no two vertices touch) map $f_\sigma : V \to \mathbb{R}^2$ that determines the position of each vertices. Then, for every edge $e = \{u, w\} \in E$, there is an injective (so that it is non self-intersecting) continuous map $f_e : [0,1] \to \mathbb{R}^2$ such that $f_e(0) = f_\sigma(u)$, $f_e(1) = f_\sigma(w)$ and $f_e((0,1)) \cap \mathrm{Im}(f_\sigma) = \emptyset$. In other words, f_e defines a non self-intersecting curve that starts at the position of u and ends at the position of w and does not cross the position of any other vertex. We further require that for any two edge $e = e'$, $\mathrm{Im}(f_e) \cap \mathrm{Im}(f_{e'}) \subseteq \mathrm{Im}(f_\sigma)$. This condition along with the definitions of f_e and $f_{e'}$ states that the curves may only cross at a common endpoint of both edges.

Seeing how long and involved this definition is, we see why we introduced this section by saying we were not going to do lots of formalism. This definition is too heavy and we will not use it in developing the theory for planar graphs. However, we will use the next theorem without proof since it helps us defining a region, which will be key in the following results.

Theorem 169 (Jordan's curve theorem)**.** *Any continuous non self-intersecting loop in the plane divides the plane in exactly two regions.*

Remark 170. If G is a planar graph and H is a subgraph, then H is planar. Pick a drawing for G and remove all the vertices and edges not in H, the result is a drawing for H. The converse does not hold as we have seen that K_5 is not planar but K_4 is a planar subgraph of K_5.

As we have said before, the number of representations is too big for us to study, therefore, we will instead study properties of drawings and especially invariants.

Definition 171 (Region)**.** Let G be a planar graph and D its drawing in $\mathbb{R}^2$. A region[16] in (G, D) is a maximal subset of the plane $S \subseteq \mathbb{R}^2$ such that S is connected in the topological sense[17] and S is disjoint from all the vertices and all the edges of G.

[16] This is sometimes called a face

[17] Out of the scope for this class but you should follow your intuition on what defines a connected region

Definition 172 (Length)**.** The length of a region R in D, denoted $\ell(R)$, is the length of the closed walk along all the edges that lie on the boundary of R.

You can see from example above that we have four regions with lengths 3, 4,3 and 6. You can try to draw different representations for this graph and count the number of regions and their lengths. You will find that there are two invariants. For any drawing D of G, the

number of regions and the sum of the lengths of the region is the same. We will continue advancing with small steps in order to prove these claims.

Definition 173 (The infinite region)**.** The infinite or outer region of a drawing D of G is the region of infinite measure[18].

[18] Again, this is out of the scope of this course, but understand it as the region of infinite size (not cardinality)

Remark 174. Observe that there exists at least one infinite region since $\mathbb{R}^2$ has infinite measure. Also, one can draw a box around a drawing for G that will contain all the edges and vertices. Since this box has finite measure, the infinite region is the one that extends outside this box.

Remark 175. For any edge $e \in E$, one can informally define a "top" region and a "bottom" region of the edge. These may be the same.

Lemma 176. *If the top and bottom regions of an edge are the same, then it must be a cut edge.*

Proof. Proof by drawing. □

Theorem 177 (Euler's formula)**.** *Let D be a drawing of $G = (V, E)$ a connected planar graph, then $|V| + \text{Reg}(D) - |E| = 2$, where $\text{Reg}(D)$ denotes the number of regions in D.*

Proof. We show this by induction on $|E|$. The base case is when G has the minimal number of edges. Since G is connected, this is when G is a tree. We know that $|V| = |E| + 1$, also, for any drawing, there can only be one region since there is no closed walk in G. We get $|V| + \text{Reg}(D) - |E| = |V| + 1 - (|V| - 1) = 2$.

Suppose that G is not a tree, then it has a cycle C. Pick an arbitrary edge $e \in E(C)$. By the contrapositive of the lemma, the regions around e are not the same, so removing it from the drawing will merge two regions, yielding $\text{Reg}(D - e) = \text{Reg}(D) - 1$. By the induction hypothesis, we also have $|V(G - e)| + \text{Reg}(D - e) - |E(G - e)| = 2$. In addition, observe that $|E(G - e)| = |E(G)| - 1$ and this implies $|V(G)| + \text{Reg}(D) - |E(G)| = 2$. □

One can easily generalize this formula (use induction) to a disconnected graph G to obtain $|V| + \text{Reg}(D) - |E| = 1 + \text{comp}(G)$. A corollary to this formula is that we can now use $\text{Reg}(G)$ to denote the number of regions in any drawing of G. Another corollary is the converse of the lemma.

Corollary 178. *If e is a cut edge of G, then e is surrounded by only one region in any drawing of G.*

Proof. The graph $G - e$ has one more component than G, so we have $|V(G)| + \text{Reg}(G) - |E(G)| = 1 + \text{comp}(G)$ and $|V(G - e)| + \text{Reg}(G -$

$e) - |E(G-e)| = |V(G)| + \text{Reg}(G-e) - (|E(G-e)| - 1) = 1 + \text{comp}(G) + 1$. Thus, we can infer that $\text{Reg}(G) = \text{Reg}(G-e)$, namely, e did not separate two different regions. □

Proposition 179. *Let G be a planar graph and D be an arbitrary plane drawing, then*

$$\sum_{R \text{ region in } D} \ell(R) = 2|E(G)|$$

Proof. We have

$$\sum_{R \text{ region in } D} \ell(R) = \sum_{e \in E(G)} \#\text{regions around } e = 2|E(G)|$$

For the second step, argue that for each edge e, if e is not a cut edge, it sees exactly two regions. If e is a cut edge, it sees the same region twice but it needs to be counted twice. The proposition trivially follows. □

Corollary 180. *Let G be a planar graph and D_1 and D_2 be two of its plane drawings, then*

$$\sum_{R \text{ region in } D_1} \ell(R) = \sum_{R \text{ region in } D_2} \ell(R)$$

Theorem 181. *Let $G = (V, E)$ be a simple planar graph with $n = |V| \geq 3$, $m = |E|$ and $f = \text{Reg}(G)$, then $m \leq 3n - 6$.*

Corollary 182. *K_5 is not planar.*

This is the main result that we will use to prove that graphs are not planar, but before proving it, we need a lemma.

Lemma 183. *Let G be a connected planar graph with $n \geq 3$ vertices. For any drawing D and any region R in D, $\ell(R) \geq 3$.*

Proof. There is no isolated edges or vertices in G but the only way for a region to have length smaller than 3 is to have a closed walk of length 2 or 1 on its boundary. This would imply that only one edge or no edge at all be on the boundary of R, this contradicts our first statement. □

Proof of theorem. Without loss of generality, assume that G is connected (if not, you can add edges and while keeping the upper bound true). Using the lemma and the last proposition, we have $3f \leq \sum \ell(R) = 2m$, implying that $f \leq \frac{2m}{3}$. Now, we can plug these values in Euler's formula:

$$\begin{aligned} n + f + m &\leq n + \frac{2m}{3} - m \\ 2 &\leq n - \frac{m}{3} \\ m &\leq 3n - 6 \end{aligned}$$

□

This is not a sufficient condition and in fact, we can easily construct graphs that satisfy it without being planar. For example, add an isolated vertices to K_5, the inequality does not hold anymore but clearly, the graph is not planar. We could also use this formula to check if all subgraphs satisfy it and that example would not work. Although it requires more work, one can still fool this method of discovering non-planar graph. Consider K_5, replace each edge with a path of length 2 (in other words, add a middle vertex to each edge). All the subgraphs will pass the test but we can still see that this graph cannot be planar since K_5 is not.

We constructed to non-planar graph from K_5 but our only proof of their planarity was to say that it was obvious from the drawing. We develop some formal tools to work with that.

Definition 184 (Subdivision)**.** Let $G = (V, E)$ be a planar graph and $e = \{u, w\} \in E$. The **subdivision** of e in G is defined as the graph $G' = (V', E')$ where $V' = V \cup \{x\}$ and $E' = E \setminus \{e\} \cup \{\{v, x\}, \{x, w\}\}$[19]. For any graph H, we say that G contains a **subdivision** of H if there is a sequence $H_0, \dots, H_\ell$ with $H_0 = H$, $H_\ell \subseteq G$ and where for each i, H_i is a subdivision of some edge in H_{i-1}.

[19] We will denote $G' = G \operatorname{sbd} e$. Also, observe that all the vertices in G keep their degree in G'

The statements we said were obvious just above follow from the following lemma.

Lemma 185. *G is planar if and only if $G \operatorname{sbd} e$ is planar.*

Definition 186 (Complete bipartite graph)**.** The complete bipartite graphs denoted $K_{a,b}$ are graphs with one empty graph on a vertices connected with all the edges to an empty graph on b vertices. Clearly, $K_{1,b}$ is a tree so it is planar. Moreover, $K_{2,b}$ can be drawn on the plane by first drawing $K_{1,b}$ as a tree and then drawing a new vertex on the leaves' side, connecting it to all the leaves. If $a, b \geq 3$, $K_{a,b}$ is not planar, although they pass the test of Theorem 173.

Since $K_{3,3}$ will be important, we do as for K_5 and prove it is not planar.

Proposition 187. *$K_{3,3}$ is not planar.*

Proof. $K_{3,3}$ is bipartite, so it does not contain any odd cycle. In particular, it is triangle free, implying that for any region R in any of its drawing, we have $\ell(R) \geq 4$. Thus, we have $2|E| = \sum \ell(R) \geq 4f$, where f denotes the number of regions again[20]. Suppose $K_{3,3}$ is planar, by Euler's formula, we have $2 = n + f - m \leq n - \frac{m}{2} \Leftrightarrow m \leq 2n - 4$. Since $m = 9$ and $n = 6$, this is a contradiction. □

[20] In the following, we will use f to denote the number of regions, m to denote the number of edges and n to denote the number of vertices

Although it is really important, the following theorem has a really long proof which we will not give in class.

Theorem 188 (Kuratowski)**.** *G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

Next, we give a definition related to subdivisions.

Definition 189 (Minors)**.** Let $G = (V, E)$ be a planar graph and $e = \{u, w\} \in E$. Define $G' = (V', E')$ with $V' = V \setminus \{u, w\} \cup \{x\}$ and $E' = E \setminus \{e \in E \mid e \cap \{u, w\} \neq \emptyset\} \cup \{\{u, x\} \mid \{u, v\} \in E \text{ or } \{u, w\} \in E\}$. In other words, we contract e into a single vertex x that we make adjacent to the neighbors of v and w. We call G' the contraction of e in G. Moreover, we say that G contains a graph H as a minor if there is a sequence $G_0, \ldots, G_\ell$ with $G_0 = G$, $G_\ell = H$ and where for each i, G_i is a contraction of G_{i-1} (we also say that H is a minor of G).

Proposition 190. *Let G be a planar graph, $e \in E$ an edge and G' be the contraction of e, then G' is planar.*

Proof. Take any drawing of G, put the new vertex on the midpoint of e and pull all the edges coming at the endpoints of e along e, until they connect with x. Since there are finitely many edges, we can do this and end up with a plane drawing for G'.[21] □

[21] This proof is not really rigorous but as we said multiple times, a rigorous proof needs some theory that is out of the scope of this class

This proposition implies that if G is planar, then any minor of G must be planar. Hence, we can show the Petersen graph is not planar by showing it contains K_5 as a minor. We will see that this is links to Kuratowski's theorem in a really nice way.

Theorem 191 (Kuratowski-Wagner)**.** *G is planar if and only if G does not contain K_5 nor $K_{3,3}$ as a minor.*

Again, we will not do the proof of this theorem in class and we will go straight to the last section of this course.

Colorings of planar graphs

Recall the map coloring problem. Given a political map, is it possible to color it using only four colors such that no adjacent territories are colored with the same color. We can formulate this in a more formal way, using what we have learned until this point. Given a planar graph G with a drawing D, is it possible to color the regions using only four colors such that no regions of the same color share a boundary edge.

In order to use what we already know from vertex colorings, we will reformulate once again. Construct a graph $G^* = (V^*, E^*)$ where each region is represented by a vertex and regions sharing a border

are adjacent. We want to know if $\chi(G^*) \leq 4$. First, we observe that G^* is also planar. Second, we give a theorem that makes the following proofs easier, however we do not give the proof of it.

Theorem 192. *Every planar graph can be drawn using straight lines only.*

Next, our goal is to introduce the beast that is the four color theorem. The first proof of this theorem was the first accepted proof that used computers. In short, they considered above a thousand of different configurations that were verified computationally and then deduced the result for any planar graph. Although, the number of different cases that need to be verified was greatly reduced over the years, the proof is still quite arduous, so we will not state it here.

Theorem 193 (Four color theorem)**.** *Let G be a planar graph, then $\chi(G) \leq 4$.*

Surprisingly, proving the weaker five and six color theorem is way easier and accessible to a student in this class. We start with the simplest.

Theorem 194 (Six color theorem)**.** *Let G be a planar graph, then $\chi(G) \leq 6$.*

Proof. We claim that there exists a vertex $v \in V(G)$ of degree less or equal to 5. This implies that G is 5-degenerate since removing vertices maintains the planarity. We have already seen that a 5-degenerate graph can be colored with six color, so this shows $\chi(G) \leq 6$.

It remains to prove our claim. Suppose that no vertices is of degree five or less, then $2m = \sum_{v \in V} \deg(v) \geq 6n$. This implies $m \geq 3n$, but G is planar, so we have a contradiction (recall that $m \leq 3n - 6$ is necessary). □

Theorem 195 (Five color theorem)**.** *Let G be a planar graph, then $\chi(G) \leq 5$.*

Proof. We will use induction on $n = |V|$. If $n \leq 5$, the result is trivial. Suppose that it holds up to $n - 1$ with $n \geq 6$, we start considering two general cases. If G is disconnected, then apply induction on each connected components, if $\exists v \in V, \deg(v) \leq 4$, then use induction on $G - v$ and color v with the color not already used by its neighbors. Suppose we are not in any of both cases, then by the claim in the proof of the six color theorem, there is a vertex $v \in V$ of degree 5. Let $N(v) = \{w_1, \ldots, w_5\}$ and $e_i = \{v, w_i\}$ for $i \in \{1, \ldots, 5\}$. Since G is planar, no subgraph of G is isomorphic to K_5, in particular, none of its subgraphs is isomorphic to K_6. This means that there exists $i < j$ such that $\{w_i, w_j\} \notin E$. Let G' be the contraction of e_i and e_j in G and

call x to be the vertex resulting of the contraction of v, w_i and w_j. We can use induction to color G' with five colors with a proper vertex coloring c.

Without loss of generality, $c(x) = 1$. Expand the coloring c to a coloring c' of G by keeping the same color for all the vertices in G and G'. We still need to color v, w_i and w_j. Since the neighbors of w_i and w_j were adjacent to x in G', none of them are colored with 1, hence, we can put $c'(w_i) = c'(w_j) = 1$ and still have a proper coloring. Lastly, v has five neighbors including w_i and w_j which have the same color. Thus, there is at least one color left for $c'(v)$ such that c' is still proper. □

We continue giving some important results, some with proofs, some without.

Theorem 196. *Let G be a K_5 minor-free graph, then $\chi(G) \leq 4$.*

Theorem 197. *Let G be a K_4 minor-free graph, then $\chi(G) \leq 3$.*

Theorem 198. *Let G be a K_3 minor-free graph, then $\chi(G) \leq 2$.*

Proof. Observe that K_3 is the smallest cycle, so if G contains a cycle, then it contains G as a minor. Our assumption then implies thtat G has no cycle, namely, it is a forest and it is bipartite, so $\chi(G) \leq 2$. □

Theorem 199. *Let G be a K_4 minor-free graph, then $m \leq 2n - 3$, where $m = |E|$ and $n = |V|$.*

Proof. We use induction on n. If $n = 2$, the result is trivial. Now, suppose it holds up to $n - 1$, where $n \geq 3$. If G is 3-connected, take two arbitrary vertices $u, w \in V$. There exists three vertex-disjoint paths P_1, P_2 and P_3 that join them. Also, we know from Menger's theorem that $G - u - w$ is connected. Take Q to be a path of shortest length in $G - u - w$ between two vertices $x \in V(P_1)$ and $y \in V(P_2)$. Now, clearly, the subgraph $P_1 \cup P_2 \cup P_3 \cup Q$ is a subdivision of K_4. This contradicts the fact that G is K_4 minor-free.

If G is disconnected, just use induction on each connected components and observe that adding the components together will not change the inequality. If G is connected but has a cut vertex v. Denote G_1 to be a component of $G - v$ and G_2 to be the rest. We use induction on G_1 and G_2 and then combine the inequalities to get $m \leq 2n - 6 + 2 = 2n - 4$.

Finally, if G is 2-connected and $\{u, w\} \subseteq V$ is a vertex cut, let C_1 be a connected component of $G - u - w$ and C_2 be the rest. Denote $G_1 = C_1 + u + w + \{u, w\}$ and $G_2 = C_2 + u + w + \{u, w\}$. In order to use induction on them, we need to check if G_1 and G_2 are K_4 minor-free. Without loss of generality, suppose that G_1 contains K_4

as a minor. Since $C_1 + u + w$ does not contain it (it is a subgraph of G), then there must be two paths in G_1 from u to w that are disjoint from $\{u, w\}$, this contradicts the fact that $\{u, w\}$ is a vertex cut. Now, we use induction on G_1 and G_2 and combine the results to obtain $m + 1 \leq 2n - 6 + 4 \Leftrightarrow m \leq 2n - 3$. □