

Name \_\_\_\_\_

**MATH 350: Graph Theory and Combinatorics. Fall 2016.**  
**Midterm Exam**

Monday, October 24th, 2016, 14:40-15:50

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The questions have to be answered in the booklets provided.

You can choose which two questions to answer. Indicate your choice on the front page. Only the two chosen questions will be graded. Each question is worth 50 points.

Write your answers clearly. Justify all your answers.

You can consult your notes and textbooks. Use of calculators, computers, cell-phones, etc. is not permitted.

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Problem	Your choice	Your score
1		
2		
3		
Total		

1. Let  $\mathcal{G}$  be the set of all simple graphs  $G$  with  $|V(G)| = 9$  such that
  - $G$  has three vertices of degree 3,
  - $G$  has three vertices of degree 5, and
  - $G$  has three vertices of degree 6.
- a) Construct a graph  $G \in \mathcal{G}$  that has a Hamilton cycle.
- b) Prove that every graph  $G \in \mathcal{G}$  is 2-connected.
- c) Is there a graph  $G \in \mathcal{G}$  that is bipartite?  
(If yes, construct a one. If no, prove it!)
- d) Is it true that every graph  $G \in \mathcal{G}$  has a Hamilton cycle?  
(If yes, prove it! If no, construct a graph  $G \in \mathcal{G}$  and show  $G$  has no Hamilton cycle.)

**2.** Let  $G = (V, E)$  be a simple graph. A set  $A \subseteq V$  is called *an independent set in  $G$*  if the induced subgraph  $G[A]$  contains no edge, i.e., if every edge  $e \in E$  is incident to at most one vertex in  $A$ . Define  $\alpha(G)$  to be the maximum cardinality of an independent set in  $G$ .

Suppose  $G = (V, E)$  is a simple graph. Prove that there is an integer  $\ell \leq \alpha(G)$  and a collection of paths  $P_1, P_2, \dots, P_\ell$  such that for every vertex  $v \in V$  there exists exactly one path  $P_i$ , where  $i \in \{1, \dots, \ell\}$ , with  $v \in V(P_i)$ .

3. Let  $G = (V, E)$  be a bipartite graph with parts  $A$  and  $B$ . Recall that for  $S \subseteq A$ ,

$$N_G(S) := \{x \in V \mid \{s, x\} \in E \text{ for some } s \in S\}.$$

- a) Show that if  $M$  is a matching in  $G$  and  $S \subseteq A$ , then  $|A| - |M| \geq |S| - |N_G(S)|$ .
- b) Show that  $M$  is a matching in  $G$  of the maximum size if and only if there exists some  $S_0 \subseteq A$  such that  $|A| - |M| = |S_0| - |N_G(S_0)|$ .